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THE UNIVERSITY OF ALBERTA

THE RELATIONSHIP BETWEEN BASAL MEASUREMENT ABILITY AND RATIONAL NUMBER LEARNING AT THREE GRADE LEVELS

by

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A THESIS

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ABSTRACT

The major purpose of this study was to determine the relationship between basal measurement abilities and rational number learning at three grade levels. A second purpose was to determine the level of performance on a test of basal measurement concepts of students in grades four, six and eight and to ascertain the effect of school, nonverbal I.Q. and sex on the measurement test scores.

A test of basal measurement concepts (TBMC) was developed and administered to 23 classes of students in grades four, six and eight in the Edmonton Public School system. Immediately following administration of the TBMC each class took up the regular rational number (fractions) unit. Fraction Achievement and Fraction Retention tests were administered upon completion of the unit.

For the first purpose three separate analyses were carried out at each grade level. First, two canonical correlation analyses were carried out using TBMC subtest scores as predictor variables and Fraction Achievement subtest scores as criterion variables then Fraction Retention subtest scores as criterion variables. One significant correlation emerged from each analysis. The most important relationship at the grade four level was linear subdivision and simple unit iteration with initial fractions. At the grade six level all TBMC subtests contributed to the relationship with linear and area subdivision showing the strongest relationship to initial fractions, comparing fractions and equivalence. At the grade eight level all TBMC subtests showed a relationship with initial fractions,

number line and multiplication/division with number line contributing most substantially to the relationship.

A second analysis involved an ANOVA on Fraction Test scores using TBMC, nonverbal I.Q. and school as factors. Results indicated that TBMC has a significant main effect at all grade levels. Nonverbal I.Q. has a significant effect at grades six and eight. School has a significant effect at the grade 4 level and on grade 6 retention scores.

A third analysis involved computing correlation coefficients among total scores for TBMC, Fraction Achievement and Fraction Retention. The coefficients were compared to coefficients between nonverbal I.Q. and the fraction tests. Results showed that the total TBMC scores are at least as good a predictor of achievement as nonverbal I.Q. at the grade four level. I.Q. is a better predictor than TBMC at the grade six and eight levels.

For the second purpose an ANOVA was carried out on TBMC scores. Nonverbal I.Q. was found to have a significant effect at all levels. School had a significant effect on grades four and eight scores. Sex had no effect on TBMC at any level.

The general hierarchy of measure development was supported by TBMC results. The more complex tasks were not mastered by students until grade eight, considerably later than their clinical counterparts. It was concluded that students in the elementary school classroom cannot be assumed to have an operational understanding of linear and area measurement tasks and models.

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THE PROBLEM

Background

The National Assessment of Educational Progress contains a report on the state of U.S. school students' fraction concepts (Carpenter et al. 1975). By the time a student has reached his thirteenth birthday he has received large amounts of instruction on fractional number concepts. Contrary to what might be expected from large amounts of exposure to fraction concepts, the end result is far from satisfactory. Not only is the junior high school student unable to operate with fractions, the assessment seems to indicate that many older students and adults are lacking fundamental skills. In the Edmonton School Systems a total of twenty weeks or more is spent on fractional number instruction. An evaluation carried out by this researcher in 1974 indicated that students in grades seven and eight had not mastered basic fraction concepts.

The concern is not new. Mathematics educators have recognized the existence of a problem with rational and fractional number learning for many decades. Kieren (1977) states that in 1831 DeMorgan wrote of his concern over the great difficulty experienced by students who were adults learning fraction concepts. DeMorgan perceived the fraction construct to be a set of computational algorithms. Research since then has tended to focus on instructional strategy, analysis of mathematical structure and sequencing, and establishing a best sequence of operations (Suydam 1969, 1973). A continuous stream of research has produced

descriptions of errors most likely to be made by students operating with fractions. Some success has accrued from these studies inasmuch as scores on achievement tests are significantly higher for one experimental group than another. The implications are usually for immediate inclusion of the "successful" practice into the curriculum. However, too often, scores for the "successful" group are well below acceptable standards. The prevalent attitude among researchers seems to be to perfect the existing curriculum rather than to change it in a fundamental way. One basis for making such a fundamental change is to study the task performance of children of various ages. Curriculum decisions could then be based on mastery of tasks at the different age levels.

A still cogent study was carried out by Washburne (1928) and directions for change were suggested in view of low achievement scores by students on various fraction concepts. A battery of tests including fractional number concepts was administered to 5,000 grade six children. Findings indicated that students who began arithmetic instruction early showed important gains in knowledge over those who started late. A later report (Washburne 1931) included the ages of students who mastered the various fraction concepts and operations. It was found that a satisfactory number (75%) of students at age ten years had mastered the operations of addition and subtraction with fractions having like denominators. When denominators differed the operations were not mastered until age thirteen years ten months.

Since the Washburne-initiated studies, a large number of research studies have been carried out (e.g., see Suydam 1964, Payne 1975), often resulting in low success on specified objectives. On the basis of

these findings several questions arise:

1. Are fraction concepts important enough for continued inclusion in the curriculum?
2. What is the nature of fraction concepts in children?
3. Could or should fraction instruction be delayed?
4. If not, why is the achievement level low — what information about children's thinking and curriculum might improve performance?

One area which could provide answers to some learning problems is that of the instructional model. Particularly in elementary school, but in junior high as well, teachers rely upon concrete models for the teaching of mathematical concepts. In the case of initial experience with fractions, three models are introduced regularly with instruction. They are set, linear measure, and area measure (region) models. The underlying belief is that the incorporation of these models into instruction will facilitate the learning of fractional number concepts. As with the use of any such model, there are certain implicit assumptions necessary for success:

1. The mathematical structure of the model is related to the mathematical structure of the concept being represented.
2. Students who are being exposed to the model as a learning device understand the mathematical structure associated with it.
3. Students are able to transfer this knowledge from the model to the abstract mathematical system under study, for our purpose the fractional numbers.

Statement of the Problem

The major purpose of this study is to explore the relationship between measurement behavior and performance on fractional number tasks. In attempting to clarify this relationship the following research

questions are studied:

1. What is the nature of the relationship between measurement concepts and rational number concepts at three grade levels?
2. Do school attended, sex, and I.Q. have an effect on measurement test scores at the three grade levels?
3. Does performance of students from different schools, having high or low measurement scores, and high or low I.Q. scores, differ on rational number tests at the three grade levels?
4. Is the measurement test a potentially useful predictor of rational number achievement at the three grade levels?

Definitions

This report contains several terms which have special meaning in relation to the study. A list of definitions of those terms follows.

measure - a unit function which assigns a number to a given quantity.

measurement - the assignment of numbers to quantities.

measurement tasks - any tasks involving measurement or properties of the measure function.

fractional numbers - the sets of equivalence classes of fractions $\frac{a}{b}$ where a and b are nonnegative integers and $b \neq o$. The term *fraction* will be used to denote either the abstract number or the numeral, its meaning dependent upon context.

model - concrete or diagrammatic embodiment of mathematical structure.

linear model - the number line.

area model - any plane surface or *region* model such as a circular or square figure which is broken into congruent parts.

rational numbers - any number that can be expressed as a quotient $\frac{a}{b}$ where a and be are integers and $b \neq o$.

Basic Assumptions

Basic assumptions underlying the present study are:

1) that linear and area measure models are two of the most common means of presenting fraction concepts;
2) that the measure concept is developmental in nature and the level of comprehension is quantifiable through administration of a paper and pencil test.

Delimitations

The following delimitations were imposed upon this study:

1. The study is delimited to grades four, six and eight from four Edmonton Public Schools.
2. The fractions unit objectives from the Elementary Mathematics Program were used as the basis for test items at the grades four and six levels.
3. Objective A-1 of the rational number unit, Junior High Mathematics Program, was the basis for test items at the grade eight level.
4. Only those objectives were included which were covered by all teachers at a given grade level.

Limitations of the Study

There are two major limitations to the study. Three different residential areas were selected as a sample of the city population. The sample does not include native or immigrant populations where English is spoken as a second language.

No attempt was made to control amount of time spent on a unit by a teacher. At the grade four level classes spent three weeks with little variation. Grade eight classes spent two to three days in review of previously learned fraction concepts and grade six, where units are longest, spent from six to eight weeks. Variance on achievement due to schools must take the time factor into account as much as possible.

Importance of the Study

Measurement models are used regularly for instruction in the elementary and junior high school. The problems associated with transfer of mathematical structure from model to abstract number system are many. It is generally assumed that such transfer takes place, i.e., that the learning of abstract concepts has been facilitated by use of the concrete material. The implicit assumption that students who are exposed to linear and area models are aware of the properties associated with those models may not be defensible, particularly in view of recent research (Steffe 1971, Montgomery 1973) concerning failure of students to operate with measure properties. If students are unable to operate successfully with measure model properties, then those related attributes of the models cannot be expected to contribute to learning of related fractional number concepts.

It has been assumed that students who possess the premeasure skills, conservation, transitivity and so on, will readily combine those skills to perform the more complex tasks of measurement. The synthesis of premeasure skills may not occur immediately and in fact may take several years to develop. Student performance on the measure test provides information as to the level of acquisition of some of these complex skills. A student's failure to understand the measure concepts may seriously inhibit initial learning with fractional numbers.

This study attempts to provide information on the development of measure concepts. It then attempts to determine the relationship between these developing concepts and achievement of fractional number concepts.

Outline of this Report

This chapter has provided a statement of the problem and questions to be answered by the study. In Chapter II a review of the literature is presented which establishes the theoretical framework for this study. The research questions are broken down into hypotheses and the statistical design outlined in Chapter III. Chapter IV provides a discussion of the development of all measuring instruments and validity and reliability measures of the completed forms. Chapters V and VI report results based on data analysis and conclusions based on those results. The original questions posed in Chapter I are addressed in Chapter VI.

Chapter II

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

Introduction

The purpose of this research was to establish the relationship that exists between the notions that underlie mathematical models used to present abstract ideas and performance by students at three grade levels on rational number tasks. The purpose of this chapter is to develop a theoretical basis for the study based on research in the field.

Although research on the topic of rationals and, more specifically, fractions, has not been neglected in the past, the focus of most research has been oriented toward algorithmic operations. It has included comparisons between various algorithmic approaches, analysis of errors made during computation, and sequencing for instruction. While the above mentioned types of research are of value to mathematics educators, they will not be discussed here as they are not pertinent to this research.

The chapter reviews three major areas of research: an analysis of the measure and rational number systems from the mathematical point of view; psychological considerations of the measure concept — mainly Piagetian-based research; and recent research which has taken the newer models of rational numbers and psychological constructs into consideration. Finally, a section is included which summarizes the theoretical framework as it emerges for the study.

The Systems of Measure and Rational Numbers

The use of measure as a model for teaching fractional numbers assumes that the mathematical structure of the measure system is similar to the structure of the fractional numbers. That this is a defensible position rests in comparing the two structures. The concern here is with the linear and area measure systems only. They represent the most commonly used instructional models, region and number line, at the elementary and junior high levels. One might question exclusion of the set model, as it is both a familiar model for teaching fractions and a measurement model. The exclusion is based on the discreteness of set elements. Unlike linear and area measures, which pertain to continuous phenomena, numerosity pertains to sets of perceptually discrete elements. Structural similarities are also greater between linear and area models. Research studies which have included both continuous and discrete fraction models have recorded difficulties which may be the result of confusion caused by the set model (Payne 1975, Owens 1976).

Measure as a mathematical model is distinguishable from the measure of everyday concern by its exactness. The functional properties of the model which are discussed below necessitate thinking in terms of an idealized situation. We need not concern ourselves with the errors associated with applications of measure systems, for example using a metrestick to measure room length gives a range of answers. We are concerned with the perfection of an ideal situation where it is not necessary to cope with error. This model provides well defined structures which facilitate the study of a less than perfect reality. It is this model of measure that is important for the present study.

Linear and area measure systems have structural similarities. It is therefore economically prudent to discuss the linear measure system and then compare the area system. This section is fashioned after Blakers (1969) and includes a discussion of only the structure relevant to this study.

A measure system is defined as a function that maps elements from a quantifiable domain space to elements of a numerical range space. In the case of linear measure we deal with a domain the elements of which are characterized by a length property. Call the domain set L. We assume that any two elements from L can be compared with respect to the length property. It follows that given $l_a \in L$ and $l_b \in L$, either l_a is the same length as l_b or l_a is not the same length. The exact possibilities are:

$l_a \sim l_b$ (if l_a is the same length as l_b)

$l_a > l_b$ (if l_a has greater length than l_b)

$l_a < l_b$ (if l_a is shorter than l_b)

The above relation on set L satisfies the symmetric, reflexive, and transitive properties and is therefore an equivalence relation on L. The equivalence relation partitions L into a set L' of distinct equivalence classes. It is possible to define a relation on L' such that:

1. $l_a' <' l_b'$ if $l_a \in l_a'$, $l_b \in l_b'$ $l_a < l_b$
2. If $l_a' <' l_b'$, $l_b' <' l_c'$ then $l_a' <' l_c'$
3. For any $l_a', l_b' \in L'$, there exists at least one l_c' such that $l_a' <' l_c' <' l_b'$

and <' is an order relation on L'. The above properties establish L'

as a densely ordered set with respect to the comparison relation <'. Furthermore, it is possible to define an operation (*) on L' whereby for any l_a', l_b' ε L' there exists an l_c' ε L' such that $l_a' * l_b' = l_c'$. The operation defines addition of lengths and says that when any element of l_a' is combined with an element of l_b' the resulting length will be equivalent to an element from another equivalence class, l_c'. Strictly speaking the combining (addition) operation is not commutative since $l_a' * l_b' \neq l_b' * l_a'$ by direct comparison. That is, direct comparison is not possible. If we define $(l_a' * l_b') \sim l_c'$ then we have the necessary comparisons for $l_b' * l_a'$ and the operation * is commutative in L'. The structure is complete, for our purpose, with addition of the Archimedian property: for l_a', l_b' ε L', then $l_a' \leqslant \eta\ l_b'$ where η is a natural number.

A structure has been established on the domain set of linear measure, L, as follows:

1. The comparison relationship < partitions L into disjoint equivalence classes, L'.
2. L' is dense with respect to the relation <'.
3. L' with relation <' is a commutative semi group under *.
4. L' with <' and * is Archimedian.

To illustrate the similarity between area measure and linear measure domains we move to an examination of area. Here we are dealing with a set of regions characterized by two dimensions. Quantification of size becomes perceptually more difficult than with length. By taking A to be the set of all regions, it is possible to establish a size relation, <, based on area, that satisfies the reflexive, symmetric, and transitive properties of an equivalence relation. The equivalence

classes form the set A'. It follows that the elements of A' are mutually exclusive. We can define an order relation, <', on A' so that for a_a', a_b', a_c' ε A':

1. $a_a' <' a_b'$ if $a_a \in a_a'$, $a_b \in a_b'$ then $a_a < a_b$
2. If $a_a' <' a_b'$, $a_b' <' a_c'$, $\Rightarrow a_a' <' a_c'$
3. for $a_a' <' a_b'$ there exists some $a_i' \in A'$ such that $a_a' <' a_i' <' a_b'$.

Analogous to the addition of lengths in L', there is an operation (*) which combines any two elements of A'. The densely ordered set A' with the operation (*) forms a commutative semi group.

The structure of A is similar to L with difficulty arising from the nature of the region elements. As two dimensional entities the comparison is more complex than simple considerations of length, a one dimensional entity. Nevertheless, it is possible to make such comparisons in the idealized situation. In reality, this problem is avoided at early grade levels by including only those domain elements which have simple, polygonal configurations that facilitate direct comparison.

It has been demonstrated that the linear and area domains possess a similar structure. Both systems have a range space which in its most complete form is the set of real numbers.[1] At beginning levels the measure model is studied as a system which utilizes only a proper subset of domain elements; those which map to the natural numbers.

Given the structure of the natural numbers, N, we know that for $\eta_a \in N$, $\eta_b \in N$ there are three possibilities for the relationship

---

[1]In fact, it cannot be shown that the entire set of reals is necessary. The positive rationals do not provide a sufficient image space but only for a subset of the domain. It is possible to map every domain element into the reals but not every real has a preimage.

between η_a and η_b: $\eta_a = \eta_b$, $\eta_a < \eta_b$, or $\eta_a > \eta_b$. It follows that:

1. $\eta_a < \eta_b$ if $\eta_b > \eta_a$
2. If $\eta_a > \eta_b$, $\eta_b > \eta_c \Rightarrow \eta_a > \eta_c$

These properties correspond to the order relation on L' but N lacks density. Clearly N is not dense with respect to the order relation since for any two natural numbers there exists either a finite number of natural numbers between them or none at all. This lack of density makes the natural numbers inadequate as a range space for the measure systems insofar as establishing a 1-1 structure preserving function is concerned.

The justification for using measure to introduce rationals rests on showing similarities between the two structures. The current study deals only with the set of fractional numbers which are isomorphic to the positive rationals. A study of the structure of the fractional numbers, F, will serve to justify use of the measure models.

F is defined in terms of N. Let N_F be the set of all $\frac{a}{b}$ such that $a, b \in N$ and $b \neq o$. Then we define the usual equivalence relation on N_F and the resulting equivalence classes define the set of fractional numbers F. Given any f_1, $f_2 \in F$, an order relation is defined on F so that if we assume $f_1 < f_2$ where $f_1 = \frac{a_1}{b_1}$ $f_2 = \frac{a_2}{b_2}$

$$\text{and } \frac{a_1}{b_1} < \frac{a_2}{b_2}$$

$$\text{then } a_1 b_2 < b_1 a_2$$

The following properties can be shown to hold:

1. $f_1 < f_2$ if $f_1 = \frac{a_1}{b_1}$, $f_2 = \frac{a_2}{b_2}$ and $a_1 b_2 < b_1 a_2$

2. If $f_1 < f_2$, $f_2 < f_3$ then $f_1 < f_3$

3. For $f_1, f_2, f_i \in F$ when $f_1 < f_2$ then
   $f_1 < f_i < f_2$ for some f_i.

The third property shows that F is dense with respect to the order relation. Furthermore, the operations of + and × can be defined so the commutative property holds and so multiplication, ×, is distributive with respect to addition, +. The set of fractional numbers, F, is a commutative semi group with respect to both operations. In particular the fractional number system with addition is structurally similar to the domain space of both the linear and area measure systems. A valid rationale for teaching rational numbers through measure models is obtained.

Transfer

It has been shown that the structure of measure is sufficiently similar to that of rational numbers to justify one assumption of using measure models. It was assumed also that students will transfer their knowledge from the models to learning rationals. There are no known studies of transfer from measure to rationals, although Sambo (1976) has initiated a project which will integrate principles of transfer into rational number instruction. His instructional design will focus on structural properties of linear and area models and their transferability to the rational number system.

Discussions in the literature of transfer vary in degree of rigour but it is generally agreed that transfer involves the use, by students,

of previously learned material in some unique way. According to Gagné (1970) there are two transfer categories. *Lateral* transfer involves the application of a learned behavior in a similar, but not exactly the same, situation. *Vertical* transfer involves the acquisition of a complex concept from simpler concepts which facilitate development of the more difficult idea. Neither type of transfer can take place unless the previous information has been acquired. Gagné states:

> It is of prime importance to note that transfer is a phenomenon which depends on previous learning. . . . something must first be learned before it can be transferred [p. 235].

Osborne (1975) states six principles of teaching for transfer predicated on identification of a prior learning set, P, and a subsequent learning set, S. The principles are:

1. The teacher must identify those attributes of learning common to P and S. These attributes include both the products of learning as well as the conditions.
2. The instructional materials and design for learning P must emphasize and/or make explicit the attributes of P that are common to P and S.
3. The more complete and thorough the learning of P, the greater the likelihood of transfer.
4. The design of instruction for S must be in terms of the attributes and conditions for learning that characterize P.
5. More powerful and inclusive concepts, principles and generalizations have greater potential for facilitating transfer than the less powerful and less inclusive.
6. Instruction for S must highlight the differences as well as the commonalities of the attributes of P and S in order to protect the learner from overgeneralization.

Osborne's focus is on instruction for transfer and therefore certain temporal conditions are placed on the transfer from P to S. In

discussing the functional properties of various measure systems Osborne points out the associations to be made between the domain and range structures. He sees these as prior learning and subsequent learning sets, respectively, when dealing *within* a given measure system. An *across* transfer problem is one in which instructional objectives involve moving from one measure system to another having common attributes. The measure system to which students are exposed first would constitute the prior learning set and that which comes second constitutes the subsequent learning set. Finally, transfer from a measure system to a mathematical system distinct from it is referred to as an *outside* transfer problem. In every context the transfer is treated as a function from the prior learning set, P, to the subsequent set, S.

Gagné's (1970) emphasis on the transfer of that which is thoroughly understood supports the supposition that students will learn rational number structure, from models having related structure, at least to some extent. The necessary condition is that the related model structure *must be understood*. Osborne's analysis suggests that more rigorous inclusion of model structure into instruction is necessary for transfer to take place. Although Osborne's model does not fit the present curriculum exactly, it does fit in a limited way. There exists a potential for developing measure concepts and applying them in the learning of rational numbers.

Psychological Research on The Measure Concept

A theoretical framework that prefaces the study of fractional numbers by understanding of measure concepts assumes that such understanding

exists in the learner. This section is a summary of research into psychological aspects of linear and area measure concepts.

Research that has focussed on the measure concept generally has been based on the work of Piaget and his associates whose initial work is outlined in *The Child's Conception of Geometry* (Piaget et al. 1960). The four stages described by Piaget in the development of measurement include:

> Stage I. Children make quantitative comparisons on the basis of single dimension perceptions. They do not conserve.
>
> Stage IIA. Comparisons are still unidimensional but attempts are made to align the objects being compared. Conservation is still not operational at age 5-6.
>
> Stage IIB. Children are beginning to conserve and so to begin measure. They are bound by a failure to compare size of units. Age 6-7.
>
> Stage IIIA. Children possess the abilities to recognize differences in unit size and number of units but not at the same time. Age 7-8.
>
> Stage IIIB. A synthesis of unit size and number of units necessary to cover the object takes place and measurement as unit iteration is operational. Age 9-10.
>
> Stage IV. Measurement development is complete, characterized by ability to calculate area on the basis of linear dimension. Prior to this the child was unable to perceive of space as an infinite and continuous set of points, a necessary condition for finding area without physical comparisons. Age 12 and older.

Before measurement becomes operational the child must internalize a number of premeasurement tasks. Not until the child is able to combine operations will he be able to measure. A great deal of research has been carried out on the prerequisites to measure, most importantly conservation and transitivity.

In reviewing the research on Piagetian-type measurement tasks Carpenter (1975) says the "pattern of development of conservation appears to be consistent" within a variety of contexts. These include perceptual illusion, perceptual distractors, transformational change, and changes in representational mode. Beilin (1964), in a study involving perceptual cognitive conflict, found that children had difficulty equating areas that were not spatially congruent. Two results from the study are of particular interest. First, children in grades K-4 were able to respond correctly when comparisons involved similar shapes but one figure had a part missing. They had difficulty with nonsimilar figures. Second, there were different responses when comparisons were made between figures composed of smaller units than the original and figures composed of larger units.

The methodology employed in studies of transitivity has greater effect on results than do those on conservation studies (Carpenter 1975). Piagetian research indicates arrival at qualitative transitivity by age 7-1/2 and quantitative acquisition about a year later. Bailey (1970) places acquisition much earlier. Bailey has suggested that children may be unable to recognize situations where transitivity is inappropriate and tend to apply transitivity consistently regardless of the problem. Braine (1959), in utilizing nonverbal techniques, placed acquisition of transitivity about two years earlier than Piaget. Smedsland (1963) discusses the sources of misinterpretation of tranitivity data and through a new series of tests places the age of acquisition of length transitivity at 8 years. Evidence suggests varying levels of performance depending upon context so that acquisition of

the transitivity concept is not easily assessed.

There is general agreement that conservation and transitivity are highly dependent and interwoven concepts. Although Piaget (1960) suggests the two develop in consequence of one another, some studies (Smedslund 1963, Steffe and Carey 1972) claim conservation appears first and Brainerd (1973) found transitivity preceding conservation. Regardless of the lack of agreement concerning acquisition of the two premeasure skills, there is general agreement that both skills must be acquired before measurement is operational.

The final measurement stage is achieved by about twelve years of age. This finding is generally agreed upon by researchers who have undertaken to validate the measurement hierarchy.

The basic element in the measure process is the unit of measure. Carpenter (1975) found that at the grade 2 level, children responded to numerical attributes as willingly as to perceptual. Difficulty arose through their failure to concentrate on two attributes simultaneously. Children's answers depended upon which attribute was being centred on. Although students were not able to logically multiply number and size of units it was found (Carpenter and Lewis 1974) that many students recognize the compensating relationship but are not able to deal successfully with it. Bailey (1974) found a similar failure to synthesize number and size of units in comparing polygonal paths. He concluded that before students can successfully compare paths they must be able to:

1) establish number relations,
2) establish length relations,
3) conserve length,

4) conserve length relations,
5) use the transitive and substitutive properties of length relations.

He concluded that the ability to establish correct relations appeared in children after age nine.

Steffe (1971) investigated, and found essential to measurement, the properties of transitivity, reversibility and substitution. He recognized a need for the children to operate with two relations simultaneously. Sinclair (1971) suggested that it is the conflict presented by focussing on different dimensions of a problem, for example number of units or size of units, that ultimately leads the child to operational measurement. The measurement ability is viewed as dependent upon resolution of these conflicts.

Montgomery (1973) carried out a study relating a child's ability to learn unit of length concepts with success in unit of area settings. The treatment involved a group using congruent units for comparison and another group using varying units. Results indicated the varying unit approach was more successful although there was no apparent interaction between success on the linear concepts and area treatments. The Montgomery study suggests that presenting students with a conflict situation will facilitate the achievement of operational measurement.

In a study of the formation of mathematical notions (Gal'perin and Georgiev 1969) it was found that all notions were subordinated by the unit concept. Instruction which used a discrete entity to define the unit was confounded by the students' failure to take size and shape into account. The authors state that defining the unit as an entity:

> . . . has two fundamental flaws. *First*, it has no relationship to the unit measure and does not develop an understanding of the fact that the number is a relationship of the quantity to the unit of measure. *Second*, such a definition permits consideration of the quantitative nature of things only when they are compared directly [p. 194].

As a result of this study a program was devised whereby units were distinguished from discrete entities and units of different sizes and shapes were used. The performance on measurement tasks of upper kindergarten children after the treatment was nearly perfect.

Minskaya (1975) carried out a study using a sample of grades one, two, and three children who were expected to move from a concrete representation of measurement to symbolic representations. Problems first took the form of everyday objects, such as a doll to be measured, a "measure" which defined the function, and the resulting number. A doll could thus have more than one resulting number depending upon the "measure" used. The task was then written in symbolic form as:

$$M_k = n \qquad \begin{array}{l} M = \text{doll} \\ k = \text{"measure"} \\ n = \text{number} \end{array}$$

From concrete situations students were expected to go to formulas where objects being measured were not present. Relationships between measure and number were established correctly by the majority of students. It is interesting that the children in this study were able to vary both unit size ("measure") and number in making comparisons.

From his studies of linear and area measure Piaget has concluded that length and area concepts develop simultaneously. Lovell et al. (1962) and Beilin and Franklin (1962) have reported that linear precedes area in acquisition. Coxford (1963) and Copeland (1974) have compiled

lists of concepts which must be attained before measurement is operational and the idea of fraction understood. A summary listing of those concepts is given in Table 1.

Research on Curriculum and Instruction of Rational Numbers

In applying measurement to real life situations and in dealing with typical school setting measurement problems, one might expect students to operate successfully by age 12 years. According to the National Assessment of Educational Progress in the U.S.A., most 9-13 year olds can make simple linear measurements but find operations where the unit must be subdivided very difficult. Nine year old children are unable to find areas when subdivided into congruent squares, and formula interpretations are impossible. Thirteen year olds are unable to compute area when presented in word problem form. Results of the NAEP on measurement indicate that students have not mastered the associated concepts by age thirteen.

Research based on the incorporation of basal measurement concepts into mathematics instruction, particularly in the area of rational numbers, is relatively recent. The research varies in nature from theoretical formulations of rational number representations (Kieren 1976) to establishment of dependencies within a hierarchy of fraction concepts (Novillis 1974). The studies reviewed here incorporate measurement into instruction on rational numbers as providing a unifying theme.

Payne (1975) presents an overview of research done at Michigan since 1968 showing a trend from algorithmic focus to integration of measure models into instruction. Green (1969), distinguishing between

TABLE 1. CONCEPTS NECESSARY FOR THE ATTAINMENT OF MEASUREMENT AND UNDERSTANDING OF FRACTION

Linear	*Area*
1. Length is determined by the configuration of material between the endpoints.	1. Conservation of area.
2. Conservation of distance.	2. Conservation of complementary area.
3. Conservation of length.	3. Unit iteration.
4. Use of rulers smaller than the object (unit iteration).	4. Dividing a region into 5ths and 6ths.
5. Accurate subdivision.	5. Multiplicative relation between length and area is understood.
6. Can use 2 or 3 perpendicular measures to locate a point.	

Fraction

1. There must be division of a whole.
2. A whole will be divided into a determinate number of parts.
3. Subdivision is exhaustive.
4. A fixed relationship exists between number of cuts of a whole and the subsequent number of parts.
5. All parts of the subdivision must be equal.
6. Fractions are parts of the whole and subdividable wholes.
7. The whole is invariant under subdivision.

an area approach and a traditional "of" approach (where the first factor defines a fractional part of some region), found:

1. The area approach was superior.
2. The area-diagram treatment worked for middle and low achievement groups
3. The area groups had difficulty with area models for multiplying fractional numbers by whole numbers.
4. All groups had difficulty with mixed numbers.
5. All groups were able to multiply fractional numbers.

Although group scores were significantly different statistically, the mean scores were not dramatically different between groups. For example, mean score on computation for the area group was 9.47 of a possible 12 and for the "of" group was 9.04.

Bohan (1970), in a study of equivalent fractions, used region and number line models to illustrate equivalence with variation in treatments due to sequencing of operations, paper folding, and use of the property of one to get equivalent fractions. Results of the study indicated a superiority of the method which incorporated paper folding.

Coburn (1973) studied the effects of the region model against the ratio model for teaching equivalence to grade 4 classes. His findings included:

1. Region diagrams were most difficult for students to interpret.
2. Region was superior on addition and subtraction of unlike fractions.
3. Achievement was higher on problem solving for the region group.

A pilot study (Payne, 1975) involving several graduate students focussed on teaching algorithms in a meaningful way through materials

and diagrams. Initial sessions ended with little success at a grade four level. The group went to a second pilot using a sample of five grade four students and including region and then set models to introduce initial fraction concepts. Number line came last. Mastery was not achieved. As a result of this pilot, a third organization of content was established in which the measurement idea was the overall theme.

Muangnapoe (1975) used an initial fraction sequence, IFS, and taught one section in his study which included five groups. Students received better scores on content taught under IFS than on other content objectives. Difficulties similar to the earlier pilot studies persisted. They were:

1) identifying a unit from a diagram,
2) realizing the need for equal size parts of a unit,
3) comparisons involving fraction symbols,
4) fractions greater than one,
5) applying fractions to the number line.

Choate (1975) studied the effects of diagrams and sequencing on learning algorithms. Achievement was found to be comparable for all groups. Transfer items were answered correctly by more students from treatments which left rule statements until last or did not give them at all.

Novillis (1976) tested grade seven students on ability to locate fractions on the number line. Control variables were length of the number line and subdivisions of the unit segment. Length of the number line was one or two units and the number of subdivisions equalled the denominator or twice the denominator of each fraction to be located.

She reports differences at the 0.001 level for both length and equivalence with students achieving higher when number line had length one and when subdivisions equalled the denominator. The interaction between factors was also significant. Novillis suggests that students treated the entire number line as a unit regardless of coordinates given.

Owens (1977), in a study of the relationship between area concept and learning fractions at the grade 3-4 level, found that:

1. Area is a significant factor in determining fraction test scores.
2. Grade level is not significant.

He concluded that area concept is related to fraction learning when area models are used during instruction. It was suggested that further research is needed on the grade level variable as it relates to the area concept.

In a discussion of the rational number construct, Kieren (1976) conjectures that in order to understand the concept of rational number students must be exposed to all interpretations. He lists seven as a basis for analysis:

1. Rational numbers are fractions which can be compared, added, subtracted, etc.
2. Rational numbers are decimal fractions and form a natural extension of whole numbers.
3. Rational numbers are equivalence classes of fractions. Thus, $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots$ is a rational number.
4. Rational numbers are numbers of the form $\frac{p}{q}$, where p, q are integers and $q \neq o$. In this form they are "ratio" numbers.
5. Rational numbers are multiplicative operators, states related by such operators or represent dilitations of the plane.

6. Rational numbers are elements of an infinite, ordered, quotient field. They are numbers of the form
$$X = \frac{p}{q}\text{, where } x \text{ satisfies } qx = p.$$

7. Rational numbers are measures, or points on a number line.

Because the natural thought of the child is not obviously related to rational number operations Kieren presents a twofold analysis. First, he presents a set of cognitive structures necessary to the development of each interpretation, and second, the kinds of instructional devices that facilitate learning within the interpretation. In a discussion of how the interpretations are related Kieren says:

> Although research is needed on the point, it seems logically and cognitively that the measurement and operator interpretations represent early direct access to rational numbers while the quotient field interpretation seems to represent a goal for later instruction [p. 48].

In a later paper Kieren (1977) draws five subconstructs from the "pool" of interpretations. The five: part whole, ratio, quotient, measure, and operator, represent distinct patterns for looking at rational numbers and form a basis for mature functioning with rational numbers.

In a colloquium paper, Sambo (1975) presents a psychomathematical development of rationals and places Kieren's seven interpretations into three categories: models; notational interpretations; and those which are fundamental to the rational number concept. Of those in the model category, he says measurement is the least abstract and most natural for children's activities.

One study has emerged from the rational number as subconstructs framework. Proceeding from his complex structure of rational numbers,

Kieren (1977) isolates the operator subconstruct for further analysis involving students at the grade 4-8 level. He identifies three stages of development and speculates on a fourth which is beyond the scope of operator situations considered in his study. Most students used a partitioning mechanism with non-unit operator problems. The ratio approach was used by only one student, a 13 year old girl, and was thought to represent the most elegant strategy for solving operator problems.

This study represents an explanation of the basis of the measure rational number subconstruct.

The Theoretical Framework

The literature review provides an outline of the systems of linear and area measure and the rational number system as a complex of representations and subconstructs. It was demonstrated that a theoretical relationship exists between the mathematical structure of the linear and area measure systems and the fractional numbers. The assumption that using measurement models embodying measurement structure, such as the number line and region models, to facilitate the learning of fraction concepts is theoretically tenable. That the theoretical relationship can be translated into a behavioral relationship has not been demonstrated.

The purpose of this study is to explore the behavioral relationship which has its basis in the above theoretical framework. Specifically, a relationship is established between measurement and fraction concepts based on student performance on sets of selected measure and fraction tasks. The study involves students at three grade levels and provides a profile of change and comparison.

Chapter III

DESIGN OF THE STUDY

Introduction

The purposes of this study were:

1) to determine how basal measurement concepts relate to achievement with rational number concepts at three grade levels, and
2) to determine the level of performance of students at three grade levels on basal measurement tasks and whether that performance is affected by residential area (school), sex and nonverbal I.Q.

The first purpose was translated into three basic questions and associated hypotheses. A threefold analysis involved factorization of the Test of Basal Measurement Concepts (TBMC) and factorization of Fraction Achievement Tests at the grade four, six, and eight levels. Factor defined subtests were then submitted to a canonical correlation analysis using TBMC subtest scores as independent variables and fraction subtest scores as dependent variables. Correlation coefficients between total TBMC and fraction test scores were obtained to provide evidence of the global relationship between the two tests. Finally, a three way analysis of variance was applied to the fractional number tests to provide an estimate of the effects of school, nonverbal I.Q., and measure ability on fractional number achievement.

The second purpose was accomplished through comparison of total scores and factors obtained from the factor analysis of the TBMC at the three grade levels. A three way analysis of variance provided evidence of school, sex and nonverbal I.Q. effects on TBMC scores.

The study was conceived as nonexperimental and correlational in nature with emphasis on relating student basal measurement abilities to learning outcomes. The study attempted to validate the theoretical relationship between basal measurement variables and rational number variables in a behavioral context. As a result of the analyses, variables worthy of future study through controlled experimentation were isolated. These variables and proposals for further studies are discussed in Chapter VI.

This chapter describes the implementation of design including statements of hypotheses tested and the associated data analyses.

Design Implementation

In view of the uniqueness of the TBMC and the desirability of establishing relationships as they exist in the classroom, a decision was made to include units of instruction on rational numbers as they are outlined by the participating school system's program for mathematics instruction. The regular classroom teachers were asked to teach each unit as usual using whatever instructional devices they thought appropriate. Classrooms which might have been in an experimental program through some other agency were excluded from taking part in the study.

The researcher was responsible for administering all tests. Immediately prior to instruction at each grade level on the rational number unit, the TBMC was administered to each class. Grade six classes began the units first and testing took place beginning mid November 1976. Grade four was tested from late November to early January and

grade eight began testing in late January. Students varied a great deal in time needed to complete the test. On the average, grade four students needed 35 minutes, grade six needed 45 and grade eight needed 40. Particularly at the grade eight level, this time varied considerably.

The rational number units of instruction were taught by the regular teacher and took from 3-4 weeks for grade four, 6-8 weeks for grade six and 2-3 days for grade eight. Some variation within grade level was due to a failure of some teachers to include all objectives included in the program guide. The grade eight unit on rationals was restricted to the first objective, a review of fraction operations, for the present study.

To monitor differences in instruction between classes, specifically the use of region and linear models, an observation schedule was used at regular intervals during the time the unit was being taught. For this purpose, each classroom at the grade four and six levels was visited an average of once a week, either by the researcher or a trained graduate assistant. The grade eight classes were not visited because review lesson presentations were strictly symbolic. A description of the observation schedule including dimensions and coding is presented in Chapter IV.

Immediately following the unit of instruction an achievement test on rational number concepts was administered to each class by the researcher. The tests were identical across schools at a grade level and included only those items which all classes had taken up during the unit of instruction. In the case of grade four classes, some teachers indicated their students had covered objectives, for example addition of

fractions, which were generally considered too difficult for that level. Items measuring attainment of addition concepts were included on the test and students were encouraged to answer them if they had been exposed to instruction on the objective. The addition items were deleted from the final analysis for the sake of across-school comparisons and factor analysis of the test. The test took a single class period to administer. A few students required time beyond the allotted class period. In all cases they were permitted to stay the extra minutes to finish.

Three weeks following the achievement test a retention test, a parallel form of the achievement test, was administered to all classes. Teachers were asked not to review for the retention test. Time needed to complete the test increased slightly for grade four and six classes, but remained the same for grade eight. Copies of all tests are included in Appendix A.

Following administration of the retention tests, I.Q. scores were collected from cumulative files on each student. I.Q. scores consisted of Lorge Thorndike verbal and nonverbal standardized scores at the grade six and eight levels and Canadian Cognitive Abilities verbal, nonverbal and quantitative scores at the grade four level. Students who had failed to take an I.Q. test at the regularly scheduled group testing session but had been tested individually, for example by a counsellor administering a WISC scale, were deleted from the study. The final choice of I.Q. scores used in the analyses was based on correlations between the I.Q. factor scores and achievement test scores. Nonverbal I.Q. scores generally correlated most highly with achievement for grades six and eight. For consistency across analyses, nonverbal

I.Q. scores were selected for analyses at all grade levels. The correlation coefficients are illustrated and discussed again in Chapter V.

The above data were gathered, and data collection instruments designed, with priority given to consistency within grade level. Although identical analyses were carried out at the three grade levels, comparisons are limited by achievement test item selection which reflects objectives for a particular grade level.

Sample

A request was made for schools using the regular program of instruction. These schools were to represent three residential areas of the city without including large numbers of native or second language students at any one level. Care was taken to request that elementary and junior high schools in the same residential category be located as close together as possible to ensure that students were representative of the same population. All elementary schools included in the study were feeder schools for junior high schools in the study. The researcher was assigned seven schools from the Edmonton Public System including one combined elementary/junior high.

One residential area having a majority of heads of household employed in a professional capacity was selected. The associated school population was too small to provide a sufficient N for analysis so a second residential area located in the same general part of the city was selected to supplement the first. The second area is considered to be similar to the first in terms of population parameters and age of community. Both areas are less than fifteen years old and are strictly

residential. The areas are combined to form level one when the "school" factor is used in analysis.

A second residential area type consists of a slightly older community of mixed occupational levels. Houses range in age from fifteen to twenty-five years and are predominantly single family dwellings. Some light industry is located on the perimeter of area two.

The third residential area consists of housing developments equal in age to the second area plus a large number classified as "older" (more than twenty-five years of age). The area is characterized by large amounts of light industry and is bounded on the south and west by the railroad yards. Occupational levels are mainly blue collar and trades.

The original sample included 169 grade four students, 202 grade six and 294 grade eight. Of these, complete data were available for analysis on 109 grade four, 137 grade six and 146 grade eight students. Most deletions were due to absence from tests or insufficient I.Q. data (mainly, at the grade four and six levels). Three entire grade eight classes were deleted for failure to take a test at the allotted time. Ten grade four students were deleted because they were in a seven year program and fifteen grade six students were deleted for similar program inequities.

The school factor was categorized on the basis of residential area location. Clearly, other variables associated with the school could have a systematic effect on performance scores. Instructional variables could differ from school to school, influencing student achievement on tests. Of particular importance to this study was the incidence of model

use in teaching rational number concepts. Each classroom was visited at regular intervals and a profile based on model use drawn up of each situation. During the visits an observation schedule was employed which enabled a quantification of the time spent using the region and linear models to illustrate various unit objectives. Observation schedule data were reduced to graphical profiles for each class. The profiles are presented in Figures 1A-1D and 2A-2G at the end of this section.

Grade Four

At the grade four level four schools are included. Schools one and two form level one of the school category. Classes from level one are designated 4A, 4B and 4C. One school forms level two and classes are designated 4D and 4E. One school forms level three with classes 4F and 4G. All classes followed the Edmonton Public School Board "Elementary Mathematics Program" unit on fractions, objectives 10-13. Textbooks used were *Elementary School Mathematics* Book 4 and *Seeing Through Arithmetic 4* (STA) as a supplementary text. Teachers were divided as to whether to teach addition of fractions at this level and some did not. It was generally felt that the concepts "greater than" (>) and "less than" (<) were beyond the abilities of grade four students although certain teachers included them. All teachers felt the number line was not useful as a teaching model but all used it as specificed in the program guide.

During instruction on fractional number concepts, grade four teachers all used region and linear models to some extent. Most used manipulative models such as paper folding and coloring to illustrate

initial concepts and equivalence. Each classroom was visited weekly and the use of models recorded. (See Figures 1A-1D for graphical profiles of grade four model use.) Classes visited only three times were those where addition was not included in the unit and instruction took less than four weeks.

Grade Six

The grade 6 classes in the study followed the Edmonton Public School Board "Elementary Mathematics Program" booklet F objectives 5-11. *Elementary School Math* book 6 was the preferred text with emphasis on exercise sheets and supplementary use of STA. Teachers at this level felt most students were capable of handling the concepts at each objective level. Once again, the number line was not viewed as a helpful device in teaching fractional concepts. Use of area and linear models varied but no teacher used manipulative materials. Schools were assigned to residential area categories in the same way as grade four schools. (See Figures 2A-2G for graphical profiles of model use at the grade six level.)

Grade Eight

Grade eight teachers did not use models to teach fractions. The rational number unit had one objective devoted to positive fractions and that was a review objective. Teachers considered the symbolic presentation adequate even in cases where students did not exhibit mastery of the concepts.

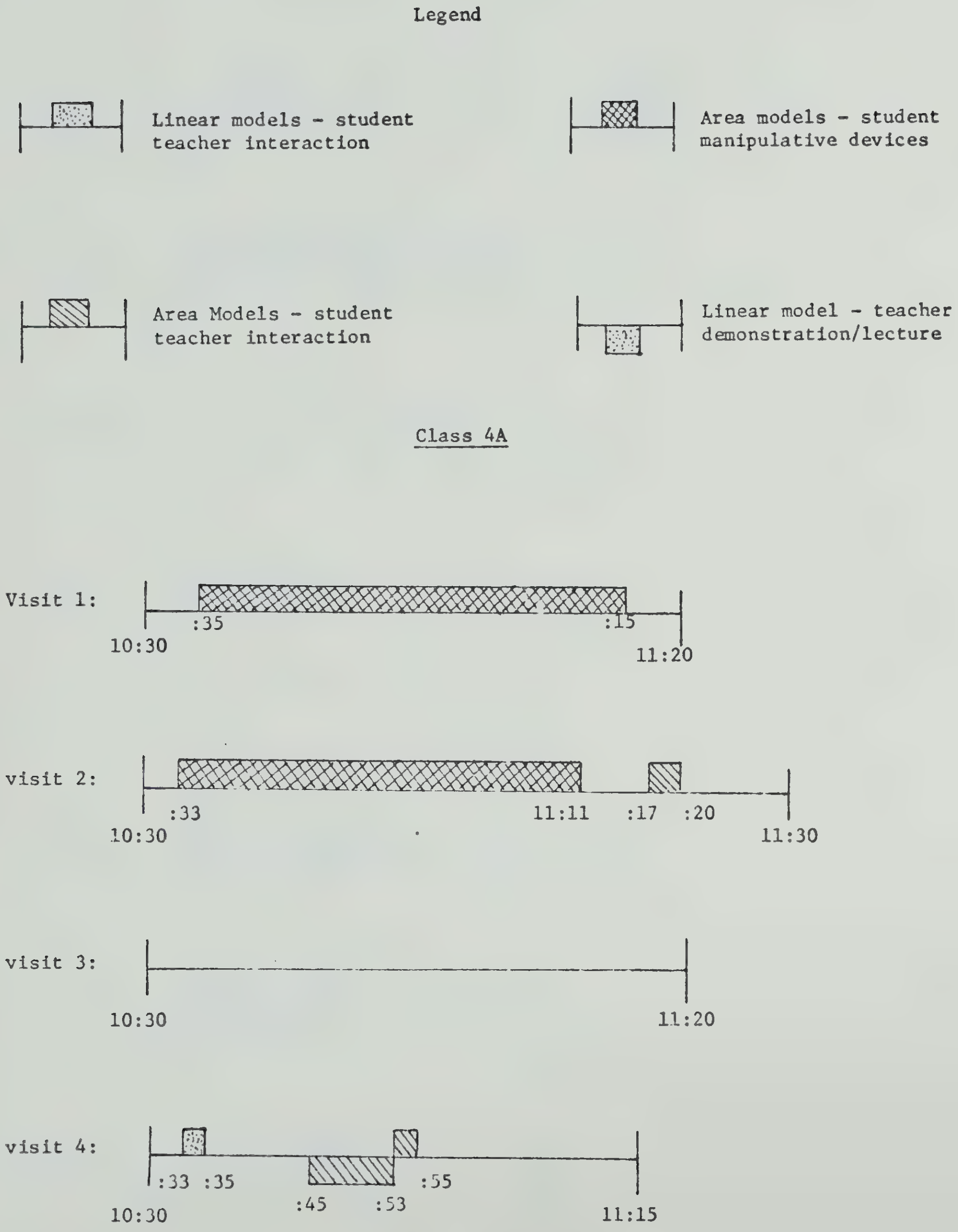


Fig. 1A. Profiles of model use at the grade four level (class 4A).

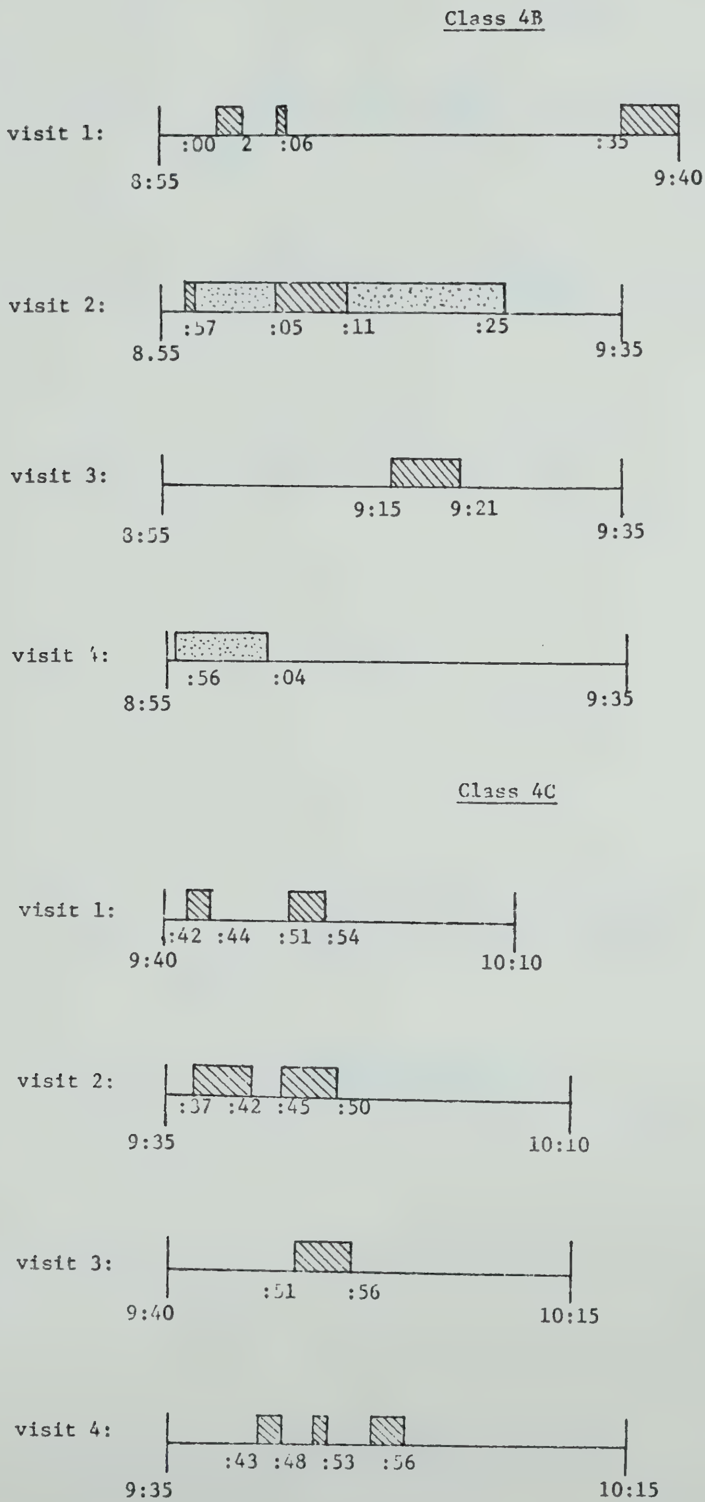


Fig. 1B. Classes 4B and 4C.

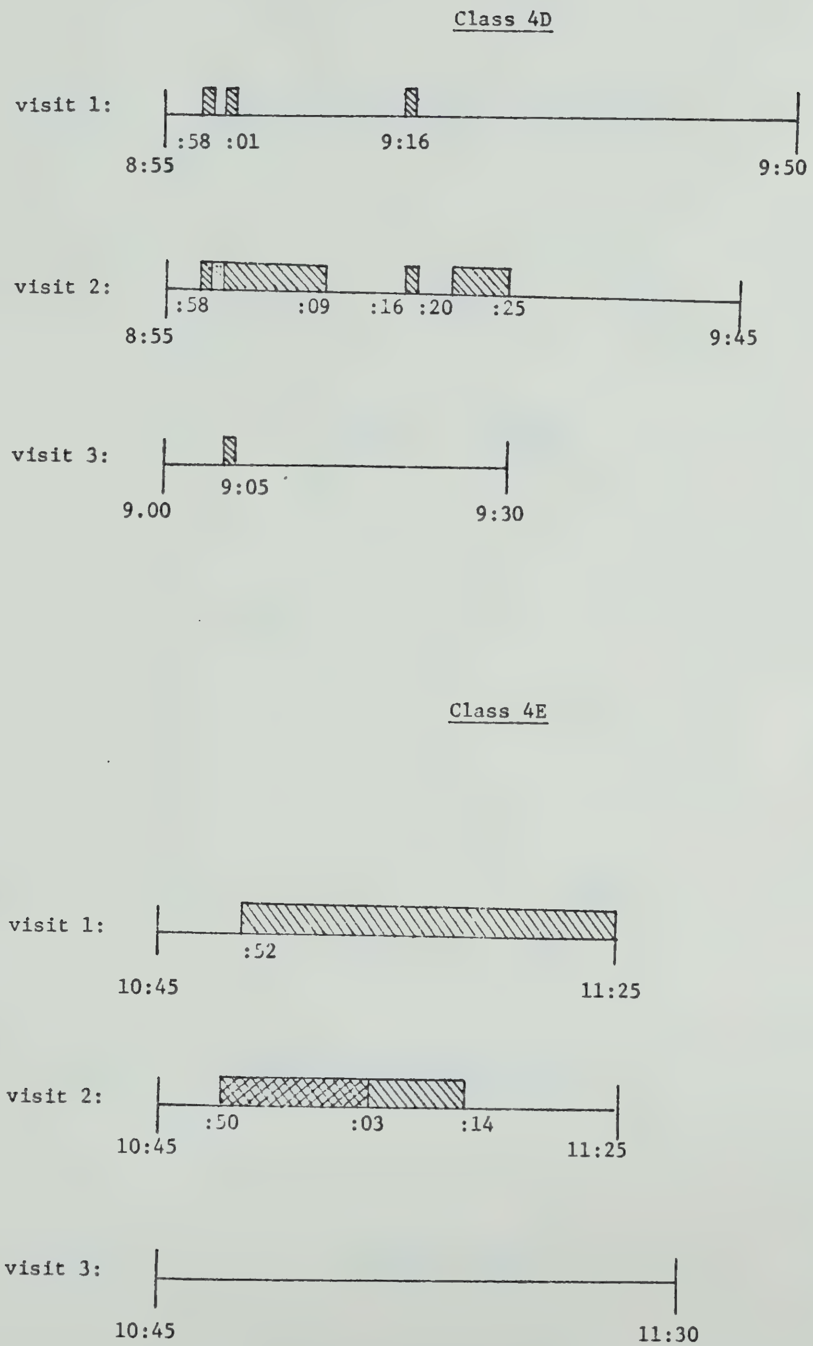
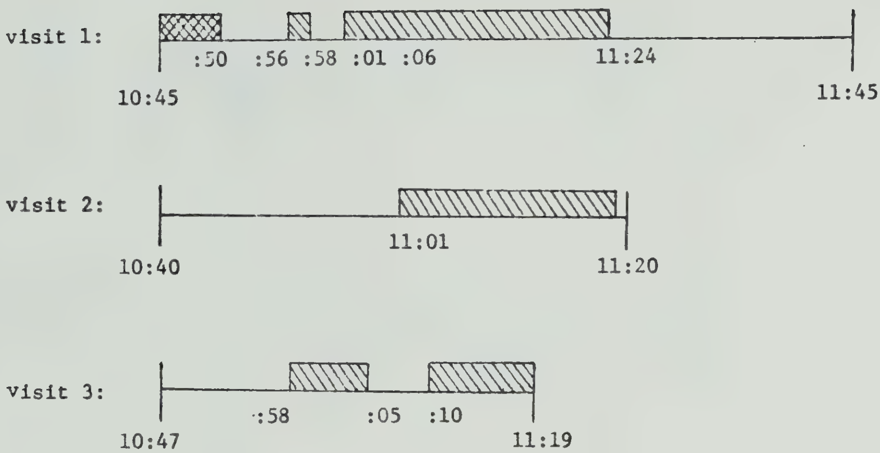


Fig. 1C. Classes 4D and 4E.

Class 4F



Class 4G

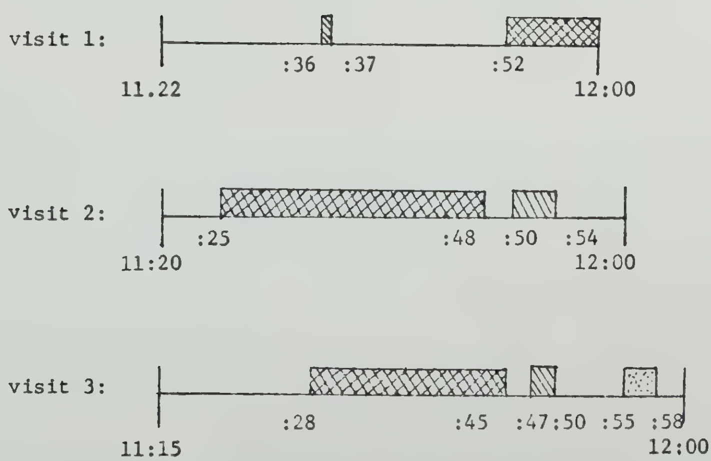


Fig. 1D. Classes 4F and 4G.

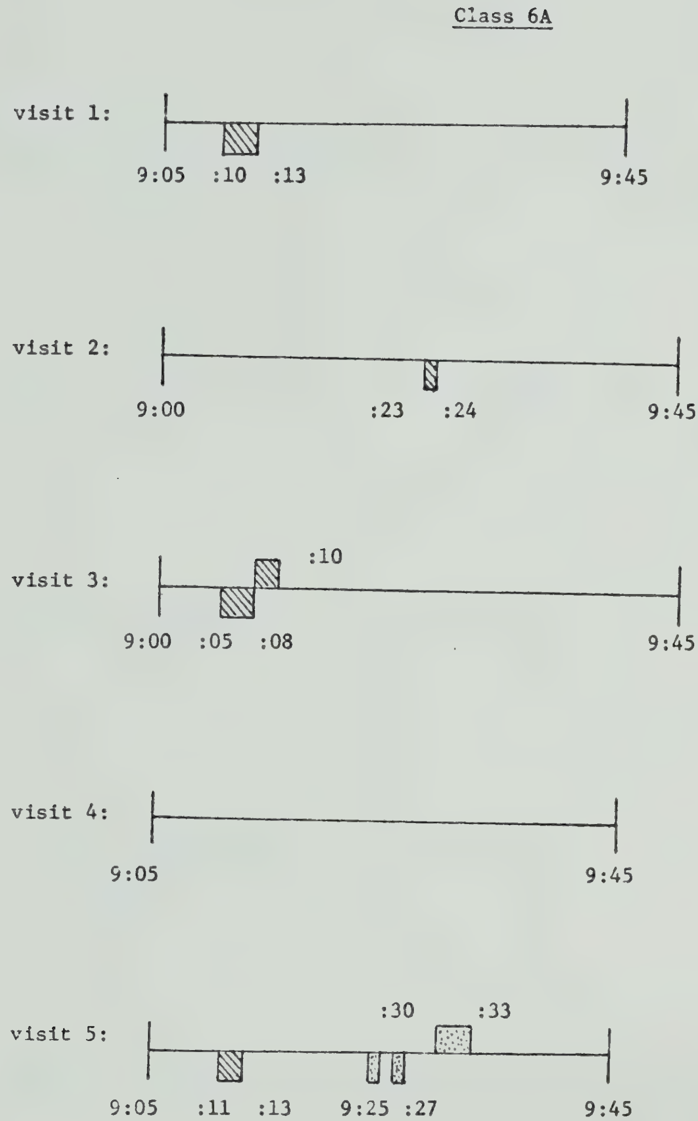


Fig. 2A. Profiles of model use at the grade six level (class 6A).

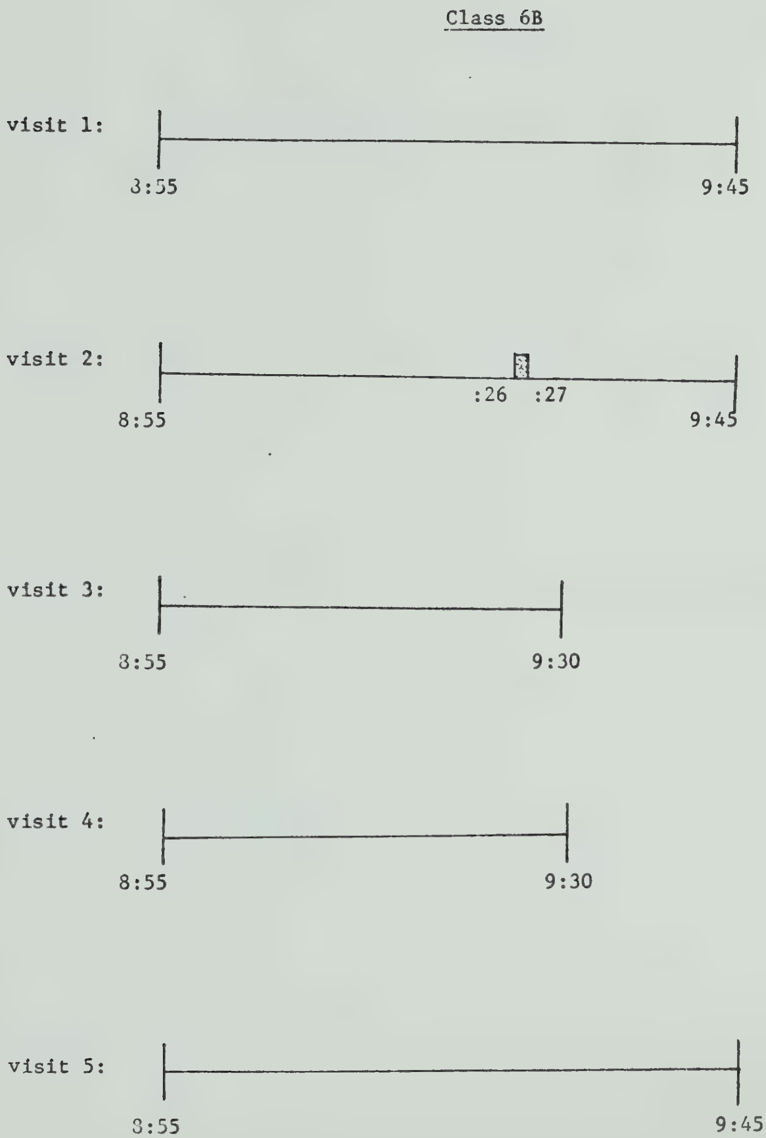


Fig. 2B. Class 6B.

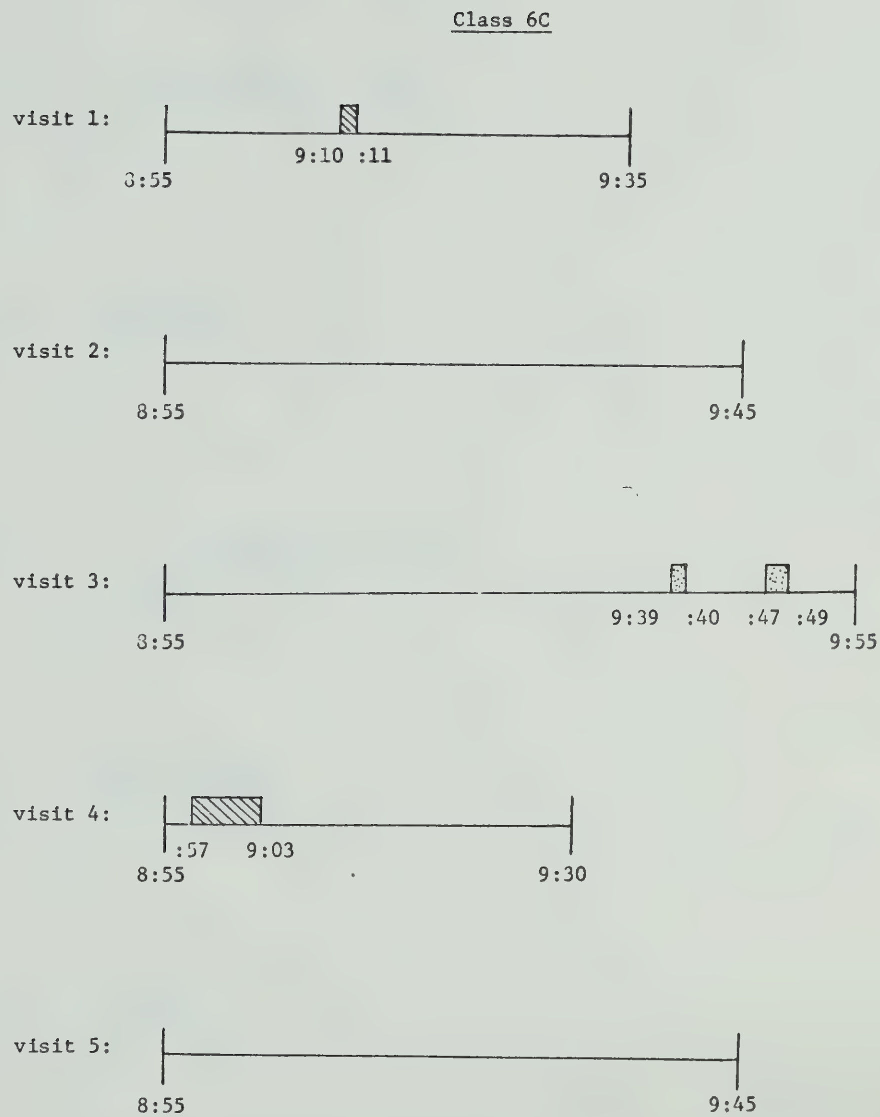


Fig. 2C. Class 6C.

Class 6D

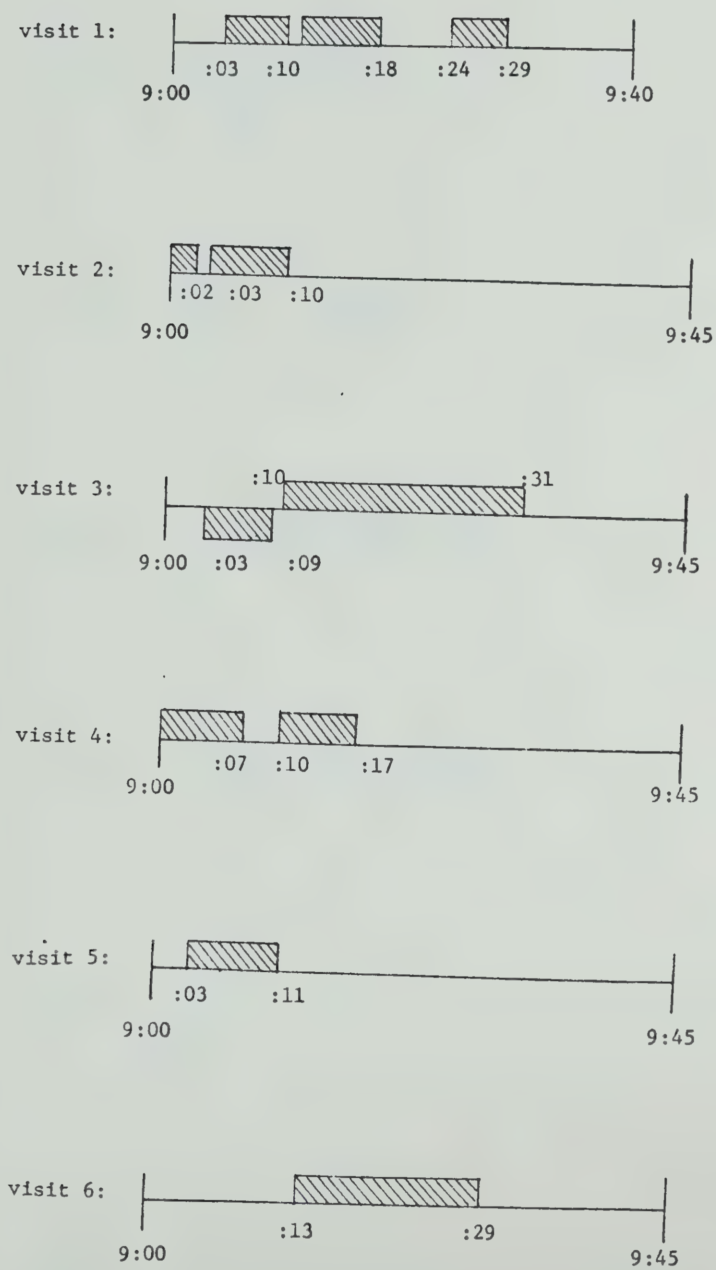


Fig. 2D. Class 6D.

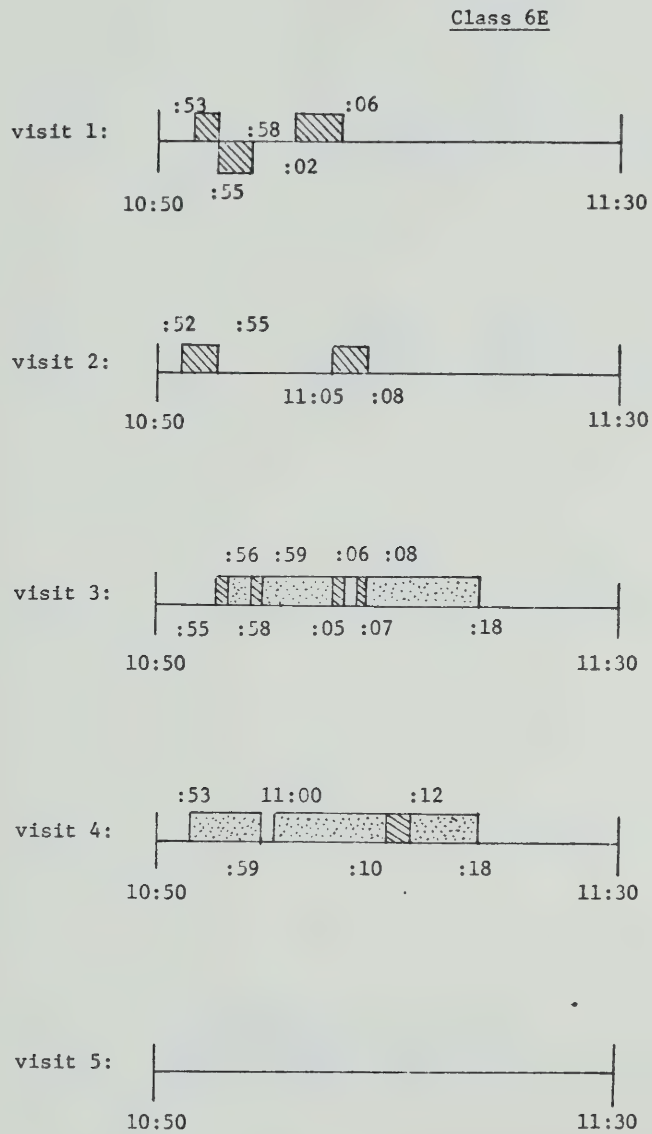


Fig. 2E. Class 6E.

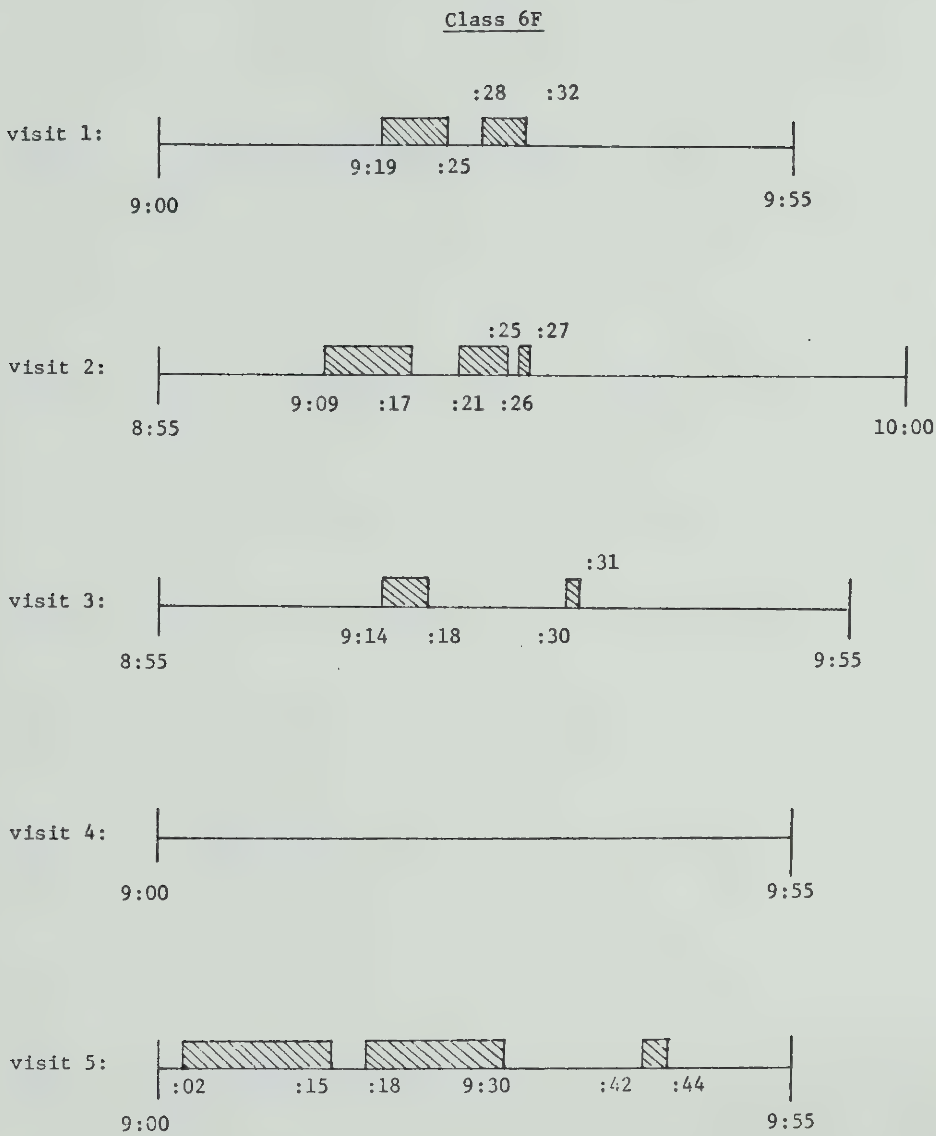


Fig. 2F. Class 6F.

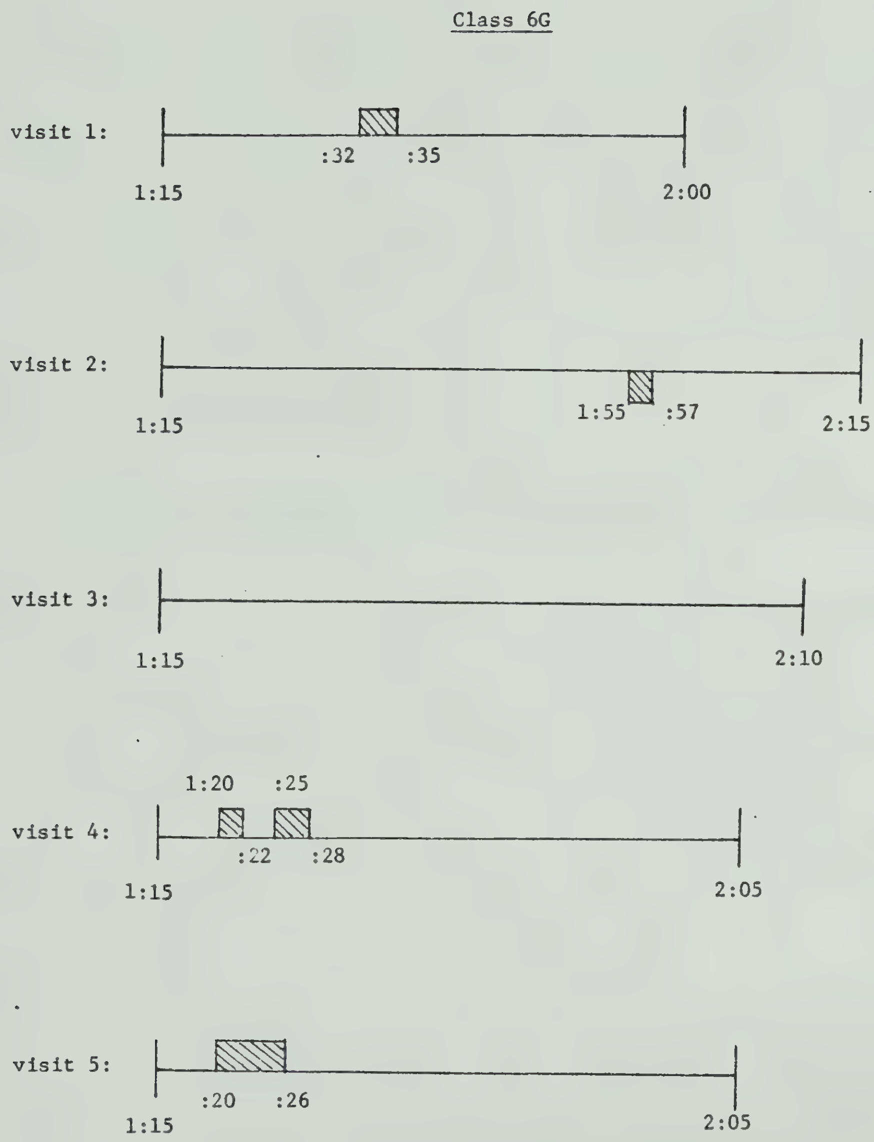


Fig. 2G. Class 6G.

Hypotheses to be Tested and Statistical Procedures

Four questions were posed in Chapter I as sufficient for the purposes of this study. In this section, the questions are translated into hypotheses which have been tested. The format for this section includes a restatement of each question, statement of the hypothesis associated with the question, and a description of the statistical process used to test each hypothesis.

The first question restated is:

I. *What is the nature of the relationship between measurement concepts and rational number concepts at the three grade levels?*

The approach to the above question involved a factorization of TBMC items and separate factorizations of items from the two rational number tests. A principal component analysis was employed to include partitioning of total variance into components. The initial structure is unique and orthogonal with successive components representing successively decreasing amounts of variance (Harman 1968). Subjecting the initial structure to a normal varimax rotation simplified the structure while satisfying the "ultimate criterion" of factorial invariance (Kaiser 1958). In view of the particular analysis, it was assumed that items which loaded highly on a single common factor and near zero on other factors constituted a subtest of the test being analyzed. Subtests as defined were then chosen as variables for further analysis but only if they contained at least four items.

On the basis of the above, scores from the TBMC subtests were treated as predictor variables and subtest scores from the rational

number tests were treated as criterion variables and submitted to a canonical correlation analysis to test the hypothesis:

Hypothesis I: *TBMC subtest scores are independent of fraction subtest scores at the three grade levels:*

a) for the Fraction Achievement subtests as criterion variables;

b) for the Fraction Retention subtests as criterion variables.

Two analyses were performed at each grade level, first using TBMC and rational number *fraction achievement* subtest scores and second, using TBMC and rational number *fraction retention* subtest scores.

In the event that one canonical correlation was significant, the hypothesis was rejected and the associated regression equations scrutinized to determine relative contributions of the different variables to the variance accounted for by the canonical correlation. In this analysis, the coefficients associated with each variable were interpreted as valid weightings since the linear combinations of variables on either side of the equation were chosen for their orthogonality as defined by the principal components analysis.

Comparisons were made of the relationships between grade levels.

Question II. *Do school attended, sex, and nonverbal I.Q. have an effect on measurement test (TBMC) scores at the three grade levels?*

It is conceivable, in view of the developmental nature of the measure concept, that certain variables might prove influential at one level but not another in the effect on measurement scores. For example, one might expect differences attributable to nonverbal I.Q. at the grade four level but not at the grade eight level. Employing a 3-way

analysis of variance design, the main effects of school, sex and nonverbal I.Q. on TBMC scores as the dependent variable were analyzed. The following hypothesis was tested:

> Hypothesis II: *There is no significant difference between mean scores on the TBMC:*
>
> *a) for the three school levels;*
>
> *b) for the sex categories;*
>
> *c) for the high/low nonverbal I.Q. categories.*

Where significant main effects were found for the school factor, the Scheffé method of multiple comparisons was employed to determine which of the factor levels were significantly different. In cases where interaction effects were significant, the estimated portion of variance accounted for was too small to justify further statistical testing of cell means. The significance of main effects was compared at the three grade levels as well as the importance of each, as determined by the estimated portion of variance accounted for by the factor, at each grade level.

In all cases where ANOVA was used cell frequencies were unequal, the most extreme case ranging from 3 to 32 students. This indicates relationships among independent variables and suggests caution be used in interpreting results. Ideally, one would have hoped for equal cell frequencies. In the present study it is clear that even had the original sample remained intact, cells would have varied greatly. In view of the existing relationships among independent variables it was decided that unequal cell frequencies would be analyzed intact rather than impose an artificiality on the sample by equalizing the n's

(Kerlinger 1968). Results of the analyses are reported with due caution in Chapter V.

In summary, the effects of school, sex, and nonverbal I.Q. on measurement were studied using a 3 × 2 × 2 analysis of variance design at the three grade levels.

Question IIa: *What is the nature of differences between measure behaviors as assessed by the TBMC, of students at the three grade levels?*

The qualitative nature of Question IIa does not admit an hypothesis but will be treated in Chapter V.

Question III: *Does performance of students from different schools having high or low TBMC scores, and high or low I.Q. scores, differ on rational number test scores at the three grade levels?*

The question was investigated by focussing on two hypotheses at each grade level:

Hypothesis IIIa: *There is no significant difference between mean Fraction Achievement scores:*

a) for the different school levels;

b) for high and low nonverbal I.Q. groups;

c) for high and low TBMC groups.

Hypothesis IIIb: *There is no significant difference between mean Fraction Retention scores:*

a) for the different school levels;

b) for high and low nonverbal I.Q. groups;

c) for high and low TBMC groups.

As with Question II, a 3 × 2 × 2 ANOVA was employed. Sample attrition due to factors mentioned earlier reduced the original sample by at least one third at each grade level. As a result, the 3 × 2 × 2 factorial design had to be modified at the grade four level. At this

level the factor "school" was reduced to two levels from three by combining schools three and four. Schools one and two were combined as usual to represent level one. The resulting factorial design was a 2 × 2 × 2. The original 3 × 2 × 2 was retained at the grade six and eight levels.

Where main effects were significant the Scheffé test for multiple comparisons was applied. Interaction effects were reported when significant but once again accounted for no meaningful portion of the variance and were put to no further statistical tests. The above hypotheses were tested also for the retention tests as dependent variables and parallel analyses carried out. Comparisons of results were made between grade level and test.

Question IV. *Is the TBMC a potentially useful predictor of rational number achievement at the three grade levels?*

The TBMC as an original instrument designed for this study should be applied in many contexts and subjected to further analysis and modification before a final form is established. In its present form, the test is correlated with rational number scores to provide a measure of predictive validity. In this study, the correlation coefficient should be viewed as a potential indication of predictive power that can be expected to increase with refinements to the measurement test. The predictive power of the TBMC was tested in part through the following hypothesis:

Hypothesis IV. *The correlation between TBMC scores and tests of rational number achievement are not significantly different from 0.*

If Hypothesis IV was rejected at any of the grade levels, then comparisons were made with nonverbal I.Q. as a traditional predictor of achievement. Predictive utility was dependent also upon the amount of variance accounted for by the relationship.

All analyses were carried out at the University of Alberta Faculty of Education Division of Education Research Services. Computer programs used in the analyses were the DEST Θ2, TEST Θ4, ANOV 35, FACT 18, and MULV Θ4.

Summary

This chapter has provided an outline of the procedures used to answer the research questions. A summary of the chronological sequence of activities associated with the research is presented as a flow diagram in Figure 3. The following chapter provides a description of the data collection instruments, their development and indices of validity and reliability.

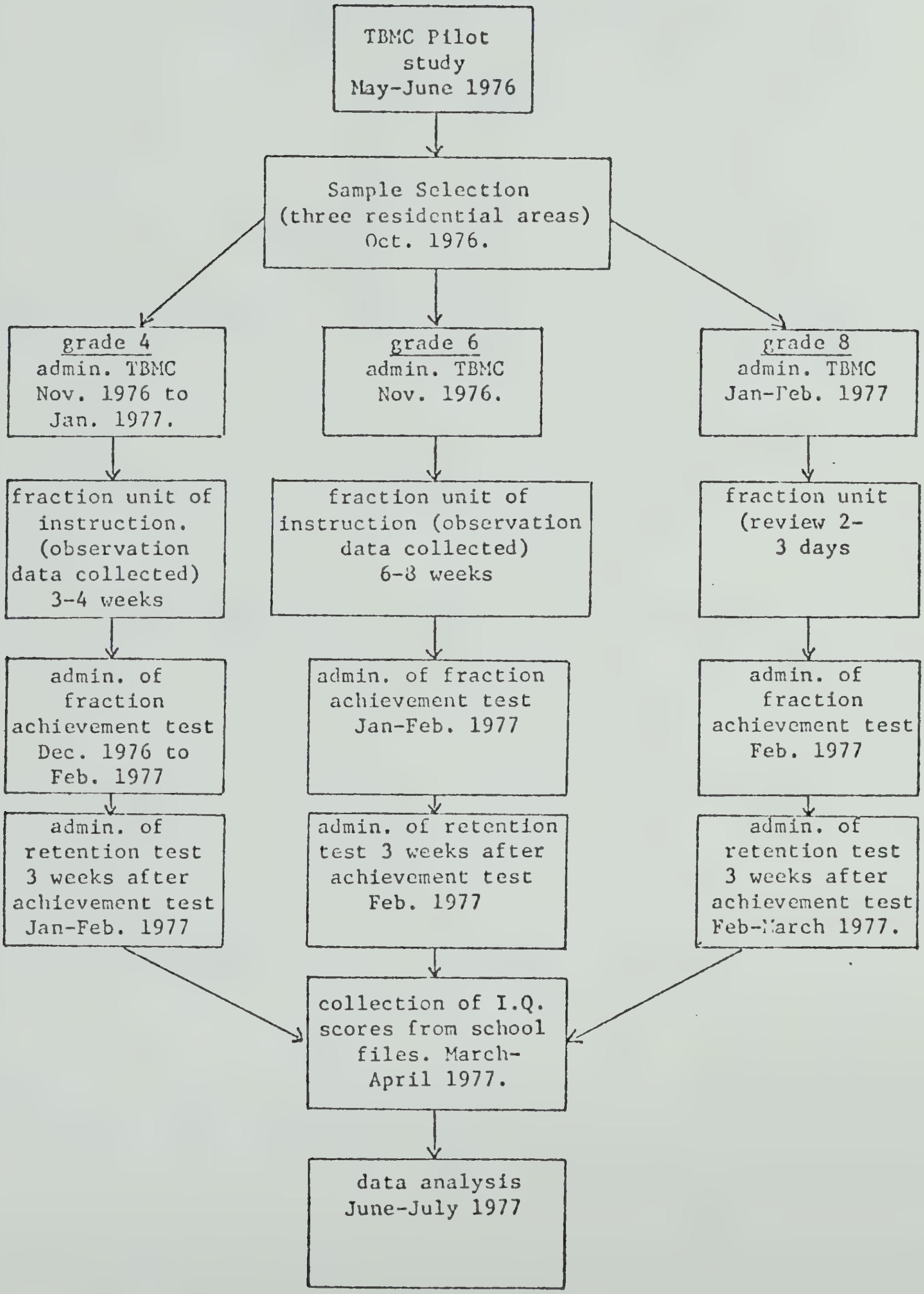


Fig. 3. Sequence of activities associated with the research.

Chapter IV

DEVELOPMENT OF THE DATA COLLECTION INSTRUMENTS

Introduction

The purposes of this study were to determine the relationship between basal measurement behavior and rational number learning and to assess student ability with basal measurement concepts at three grade levels. This chapter is concerned with the construction of data collection instruments, their validity and reliability.

Three types of measuring instruments were used to collect data. The Test of Basal Measurement Concepts (TBMC) was designed by the investigator for this study and is of particular importance to the study. A paper and pencil test of basal measurement concepts does not exist and methods of assessment in the past have been clinical in nature. They have generally involved interview settings with various representational devices which were carefully explained by the administrator on an individual basis. In the interest of presenting a test in keeping with typical classroom testing procedure, the TBMC represents a departure from the classical approach to testing measurement behavior.

In choosing test objectives from the domain of measurement concepts, it was necessary to be selective. To this end, the objectives around which test items were designed were restricted to those which reflect linear measure and area measure abilities, those properties embodied in fractional number models. The two major criteria for item selection were relevance to the study and appropriateness for inclusion

in a pencil and paper group testing situation. A great deal of attention is given to the development of the TBMC including item selection, reliability and validity measures.

Development of the rational number tests and the observation schedule, also designed by the investigator, are discussed as well.

I.Q. scores were taken from cumulative records supplied by the schools and included only those collected during the regularly scheduled group testing sessions. Nonverbal I.Q. scores were chosen for analysis with collected data due to higher correlation with fraction test scores than between verbal I.Q. and fraction tests at the grade six and eight levels. Validity and reliability of I.Q. scores are accepted as adequate.

The Test of Basal Measurement Concepts (TBMC)

Content

The TBMC was developed to assess level of performance on linear and area measure tasks. In the past, a great deal of research has been carried out on isolated measure tasks, and stages of development in children have been established. A summary of that research was given in Chapter II, Section IV. The research on measure concepts has been almost exclusively restricted to premeasure tasks in clinical settings with children aged 4 to 10. Very often mastery of certain measure tasks was attributed to children who had satisfactorily completed manipulative tasks. Less work has been done using pictorial representations and indeed it should not be assumed that children who function at an enactive level can handle more abstract levels of representation. The

present rational number program at the elementary level is based on an assumption that students are able to understand more abstract and symbolic measurement representations. The development of the measurement test was therefore based on that assumption.

The measurement test was designed as a paper and pencil test to be administered in classroom group situations by the researcher. This type of testing is in keeping with instructional materials and strategies at the levels under study.

The test was visualized as having two main components, a linear and an area subtest. The linear subtest is dealt with first as it has fewer dimensions; therefore fewer items were included. Items which assess a child's understanding of linear measure could provide a measure of his ability to deal with number line tasks. A particular instance is the student who is unable to subdivide a line segment into n equal parts because he does not appreciate the need for n-1 "cuts". Instead, he may divide the segment by placing n cuts between endpoints or, starting at one end, place n-2 cuts plus the endpoints. In one case he divides the segment into n+1 parts and in the other case into n-1 parts. It is conceivable that such a student will count endpoints when placing fractions on the number line. Another possibility, suggested by Bailey (1974 p. 524) is that students who focus on either size of subsegments or number of subsegments, but not both, are faced with perhaps unresolvable conflict when two equivalent fractions are located at the same point on the number line.

Piaget (1960) has said, in reference to measurement of length:

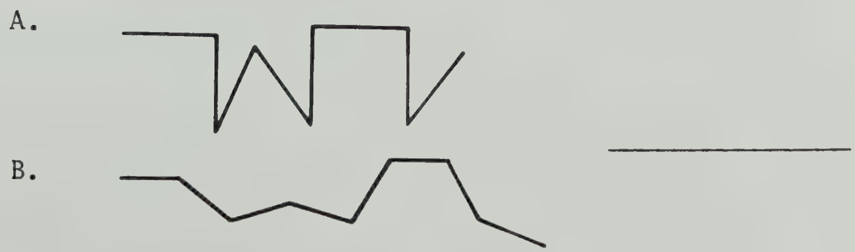
> While conservation and hence qualitative transitivity are achieved at a mean age of 7-1/2, measurement in

> its operational form (i.e., with immediate insight and not by trial and error) is only achieved at about 8 or 8-1/2 [p. 126].

It is assumed that most students from grade four and up are capable of operational measurement of length. It is assumed also, therefore, that premeasure tasks have been mastered by students at the given grade levels.

The first linear measure items are fashioned after Piaget and more recently Steffe (1971) and Bailey (1974). They are designed to assess the child's ability to compare lengths based on different sized units and different numbers of subsegments in a given segmented path. Clearly, difficulty of path comparisons is dependent upon how many attributes are varied in the task. For example, two paths may have the same size unit segments, equal numbers of segments, similar configurations, and similar extremity positions. Such comparisons would present little conflict to children and they would have little difficulty in making correct comparisons. When attributes are varied between paths, the solution is more difficult depending upon how many attributes have been varied. Items were therefore included which varied unit size and number in different path configurations. Students were asked to compare lengths of paths similar to those shown here.

BELOW ARE TWO PATHS, A AND B. IS PATH A LONGER THAN B, SHORTER THAN B, OR ARE THEY THE SAME?



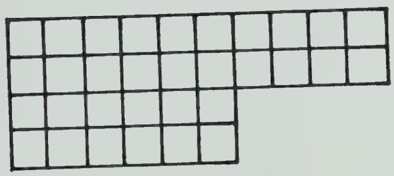
Linear measure, as a "fusion of subdivision and change of position" (Piaget 1960) may not be fully operational in a child who can otherwise make correct judgments based on intuition.

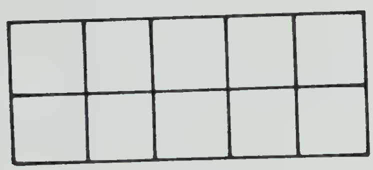
A child who cannot correctly compare paths with varying attributes may be able to isolate a single unit through trial and error and therefore have a rudimentary grasp of unit iteration. A second item type required students to use a given segment in constructing other segments two times and five times as long. The final items required students to subdivide given segments into five and six equal parts. Students who are operational in terms of unit iteration were expected to complete these tasks satisfactorily. Students not at the operational level might still be able to arrive at a correct solution through trial and error starting with a perceptual interpretation. Students at the more elementary levels might reasonably be expected to count the endpoints, or partitions, instead of the subsegment, thereby arriving at incorrect solutions. Correct answers were accepted whether they were based on trial and error or more elegant methods. These items required a lower level of cognitive ability than the comparisons of segmented paths.

The area subtest consists of items paralleling those of the linear subtest plus others which result from the two-dimensional character of area. The first set of items is based on unit iteration. Here, as with linear measure, the size of the unit, the number of units required for coverage, and the configuration of the regions being compared were varied. A sample pair is shown as follows:

BELOW ARE TWO ROOMS, A AND B. THE ROOMS HAVE TILE ON THE FLOOR. WOULD YOU HAVE MORE SPACE TO PLAY IN ROOM A OR ROOM B, OR ARE THEY THE SAME?

A.



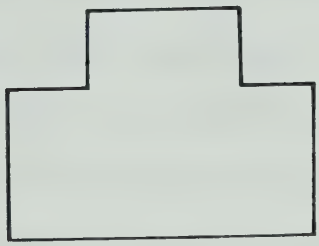


B.

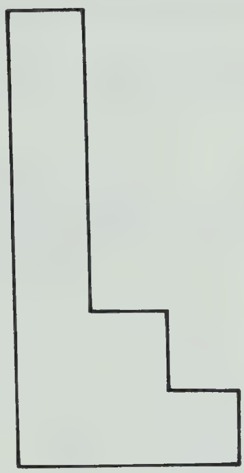
The provision of a unit grid establishes a focus for the method of solution. Students are perceptually directed to consider both unit size and number of units. Where figures are not transposed on a grid, the comparison is more difficult in that the student must pick a suitable unit of measure. This was tested with a set of items that called for comparisons of pairs of irregular polygons not subdivided into units. A sample of this item is illustrated here.

BELOW ARE TWO YARDS, A AND B, WITH FENCES AROUND THEM. TELL WHETHER MORE CHILDREN COULD FIT INTO A OR B OR WHETHER EACH YARD WOULD HOLD THE SAME NUMBER.

A.



B.



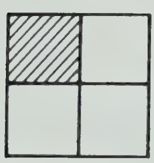
Subdivision of area figures was included since, just as with linear measure, before unit iteration can take place the quantity being measured must be subdivided. This group of items was designed to test subdivision of three different plane figures into five or six equal parts. Since this skill is theoretically acquired at the grade four level, it was believed that differential abilities might be distinguished with respect to different configurations as well as number of subdivisions. Students

were asked to subdivide a square, a circle and a rectangle into five and six equal parts.

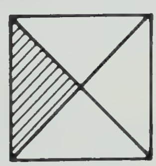
The final two sets of area measure items were ones that proved most difficult for students at all levels. One set consists of pairs of similar figures which have been subdivided and a portion shaded on each figure. Students were asked to compare the shaded portions as to size (area). A correct response can be given only when students are able to reason that since the pair of figures are congruent and each is subdivided into n equal parts, then the noncongruent parts must have equal area. Examples of this item are shown below.

LOOK AT THE PAIRS OF FIGURES BELOW. THINK OF EACH FIGURE AS A CAKE. IN EACH CASE, TELL WHICH SHADED PIECE WOULD BE BIGGER, A OR B, OR ARE THEY THE SAME?

A.



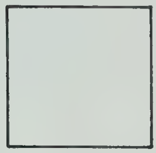
B.



The set included pairs of similar but not congruent figures, pairs of congruent figures, and one pair where subdivisions were congruent but the figures consisted of a square and a rectangle.

The last set of items represented a terminal behavior in area development not expected to be manifested until 11-12 years of age. Although not necessarily related to region models used to teach fractions, it was felt that as an indicator of complete area mastery the items would be desirable to include. One of the items is given as follows:

DRAW ANOTHER SQUARE WHICH IS TWICE AS BIG AS THE ONE SHOWN.[1]



Grade four children were not expected to complete all items satisfactorily. Nevertheless, since they are exposed to applications of the properties being tested in the unit on fractions, it seemed fair to assess their level of performance.

The TBMC Pilot

The TBMC was piloted on three grade five classes and two grade four classes in May and June of 1976. The pilot served two purposes. First, it provided information on adequacy of wording of instructions for each item and, secondly, it provided scores for item analyses on which to base item changes.

The test was revised three times on the basis of grade five pilot studies and once on the basis of a grade four pilot study. The reading problem at that level convinced the researcher that written instructions should be minimized and a testing protocol was established with an oral explanation of each item. Words used to describe the items were often those used by Piaget and included suggestions from staff and graduate students in the Department of Elementary Education at the

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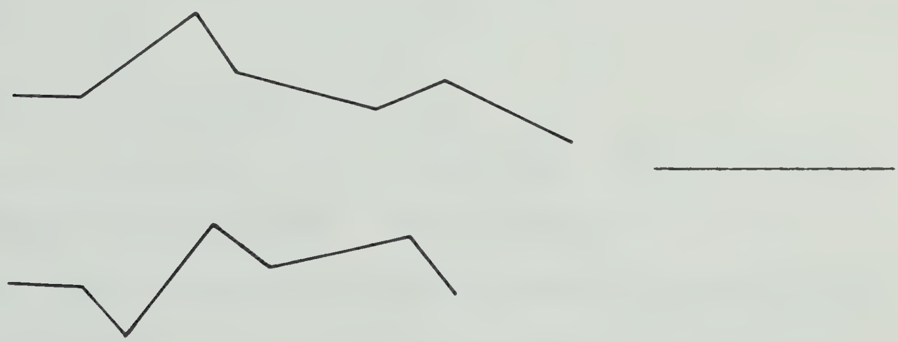
[1] These items were deleted from the final analysis due to ambiguities in response which suggested students had not understood the question.

University of Alberta. In its final form, twenty minutes were required to go over the instructions at the grade four level. Less time was required at the grade six and eight levels.

Items were altered on the basis of item difficulty and discrimination. The first administration of the test yielded several negative discrimination indices. A survey of the items with correct and incorrect responses showed that in some cases the correct answer was possible from an examination of perceptual attributes. For example, the following item was included in the first pilot. Most low scoring students responded correctly to the item while many students in the high scoring half of the class did not. It is possible that the low

A.



B.

scorers were functioning at a lower level and answered the question (comparison of length) on the basis of endpoint alone. The better students may have realized the endpoint location was not sufficient but were unable to deal with the intervening configuration. Items of this type were altered so that a perceptual clue was not available.

Generally, a discrimination index of .40 is regarded as desirable but not essential for item inclusion (Mehrens and Lehmann 1973). It is

sometimes desirable, and is the case for the TBMC, that items of extreme difficulty and therefore low discrimination be included. For example, the doubling the square item was nearly impossible at the grade four level. It was left in because it reflects an important attribute of measurement even though it is a poor discriminator at the grade six and eight levels. Items were considered acceptable when the difficulty and discrimination power were considered reasonable *in view of* the grade level at which the test was given. That is, if the difficulty index and discrimination were extremely low for items which were considered very difficult for grade four, but which were considered important in assessing level of measurement ability, then the items were included. Item statistics for the TBMC are reported in Appendix B for the three grade levels.

Validity and Reliability of the TBMC

Validity is "the extent to which a test does the job for which it is used" (Mehrens and Lehmann 1973). The validity of a test is dependent upon the use to which it is put. Generally, a test is used for one of two purposes: to predict or to describe. The first of these purposes requires criterion validity measures and the second requires evidence of content and construct validity. The measurement test is intended for predictive and descriptive purposes so evidence of all three validities is discussed.

Content validity. This type of validity tells how well the test items sample the domain to which inferences are being made. The measurement test was constructed on the basis of previous research tasks

which are well known in the literature and have the support of replication. All items on the present test were fashioned after measure tasks included by Piaget in his studies on the development of measure in children. Most of the tasks have been presented in more recent contexts by different researchers and in some cases the tasks have been revised (for example Beilen 1964, Bailey 1974, etc.). A sampling from the domain of measure for this test includes items representing linear and area measure behavior for children aged 8-1/2 and over. Table 2 provides a description of linear and area measure behaviors being assessed by the TBMC. In view of the agreement on developmental stages and definition of measure concepts found in the literature, it is concluded that the TBMC represents a valid sampling of content.

Criterion validity. Of the two types of criterion-related validity, concurrent and predictive, we are concerned here with whether scores on the TBMC are related to performance on rational number tests as the criterion. *Predictive validity* is a major concern and is expressed as a validity coefficient, the correlation coefficient between TBMC and criterion. The coefficient is dependent upon both test and criterion and therefore the criterion must satisfy the same conditions as the test. According to Mehrens and Lehmann (1973), to be adequate, a criterion must satisfy three conditions:

1) relevance — the criterion must be valid,
2) reliability,
3) freedom from bias.

The first two are discussed in the next section, Validity and Reliability of the Rational Number Tests, and are shown to be both valid and

TABLE 2. DESCRIPTION OF THE TEST OF BASAL MEASUREMENT CONCEPT ITEMS

Description of behaviors being assessed	Items
Linear measure	
simple unit iteration. Given a unit segment, construct a segment n times as long	12, 13
subdivision of a line segment	6, 7
operational measurement. Compare polygonal paths with varying units.	1, 2, 3, 4, 5
Area measure	
subdivision of regions	19, 20, 21, 22, 23, 24
operational measurement. Units are suggested by grid imposed on regions.	14, 15, 16, 17, 18
operational measurement. Compare regions. Student must provide own strategy. No units shown.	8, 9, 10, 11
subdivision comparisons given noncongruent units of subdivided regions.	27, 28, 29, 30, 31, 32
double the region (area)[1]	25, 26

[1]Deleted from final analysis.

reliable instruments. The bias referred to in number three involves knowledge of predictor scores that might affect scoring of the criterion tests. The researcher scored all tests and did not associate predictor scores with criterion scores until all scoring was completed. Sample size would tend to preclude such bias as well.

The criterion measures satisfy the three conditions and the validity coefficients are reported in Table 3. There are two criterion measures at each grade level, Fraction Achievement test scores and Fraction Retention test scores. The Pearson Product Moment correlation coefficient was the procedure used. According to Cronbach (1970), it is unusual to achieve validity coefficients greater than .60. The validity coefficients ranging from .38 to .56 are considered adequate at the three grade levels.

TABLE 3. VALIDITY COEFFICIENTS BETWEEN MEASUREMENT TEST AND TWO CRITERION MEASURES AT THREE GRADE LEVELS

Grade level	Criterion	Validity coefficient
4	achievement	.47
	retention	.56
6	achievement	.55
	retention	.53
8	achievement	.48
	retention	.38

Construct validity. Construct validity provides evidence of the contribution to test scores of the psychological constructs that a test is measuring. Construct validation requires time — it is established through "a long continued interplay between observation, reasoning and imagination" (Cronbach 1970). Statements concerning construct validity of the test in its present form suggest how well the test fits the theoretical constructs it was designed to measure. Test items were developed according to the linear and area measure concepts discussed in the first part of this chapter. Construct validity is established if it is shown that the components which contribute to score variance are analogous to certain measure theoretic concepts. A principal components analysis of the test items into common factors suffices to provide the necessary evidence.

There were thirteen item eigenvalues greater than one at the grade four level, ten greater than one at the grade six level, and eleven greater than one at the grade eight level. Six factors were extracted at each level accounting for 41.6%, 42.0% and 44.6% of the variance at the respective grade levels. The number of factors extracted was a function of interpretability. A ten-factor extraction is shown in Appendix C for each grade level, accounting for about 60% of the variance in each case. The greater number of extracted factors shows a breakdown into unique factors, a study of which is not necessary for our purpose here. Tables 4, 5 and 6 provide the six-factor solutions for the grades four, six and eight TBMC.

The usual practice of choosing factor loadings $\geqslant .33$ is appropriate here since theory validation is the primary concern. All loadings

of .33 and greater were included in Tables 4, 5 and 6. Of the six factors extracted at the grade four level, four account for more than 75% of the common variance. Test items 14-18 have high loadings on the first factor. All are area unit iteration items. Item 8 also loads heavily on factor one and is an area comparison item but without units delimited. It is an item on which many students used a grid to solve.

Factor two has four substantial loadings. Items 6 and 7 are linear subdivision and items 12 and 13 are linear items which require the construction of multiples of a given segment.

Factor three shows seven loadings greater than .33. Item 14 is an area unit iteration involving congruent units and 20, 22, 23, and 24 are subdivision of a region involving subdivision of a circle into five and six equal parts and a square and rectangle into six parts. Items 26 and 30 involve equal regions having equal but noncongruent subdivisions.

Factor four shows the five items 1-5 of segmented path comparison and 21, subdivision of a rectangle into five equal parts.

Factors five and six show three and two loadings greater than .33, respectively, but have no obvious interpretations.

Four common factors distinguish themselves at the grade four level:

1) area unit iteration,
2) subdivision of length and constructing multiples of a length,
3) subdivision of a region, and
4) linear unit iteration (operational linear measure).

TABLE 4. PRINCIPAL AXIS FACTORING WITH DIAGONAL AS INPUT UNDER NORMAL VARIMAX ROTATION FOR TBMC GRADE FOUR

Variable item	Factor 1	2	3	4	5	6	Communality h^2
1				.45		.71	.38
2				.33			.44
3				.65			.46
4				.48			.32
5				.55			.38
6		.67					.54
7		.67					.52
8	.58						.42
9							.27
10							.09
11							.33
12		.55			.53		.68
13		.57					.51
14	.45		.33				.48
15	.58						.39
16	.62						.47
17	.62						.53
18	.53						.48
19							.27
20			.52				.43
21				.44			.34
22			.44		.48	.40	.63
23			.49				.31
24			.36				.29
25							.54
26			.53				.40
27					.34		.47
28							.38
29							.35
30			.50				.40
% Total variance	8.65	8.11	7.61	6.68	5.34	5.23	
% Common variance	20.78	19.49	18.29	16.05	12.83	12.57	
Sum of communalities							12.48

Total variance accounted for 41.60

TABLE 5. PRINCIPAL AXIS FACTORING WITH DIAGONAL AS INPUT UNDER NORMAL VARIMAX ROTATION FOR TBMC GRADE SIX

Variable item	Factor 1	2	3	4	5	6	Communality h^2
1			.62				.47
2							.18
3			.60				.37
4			.39				.36
5			.50				.52
6		.64					.53
7		.71					.56
8	.61						.44
9						.41	.25
10					.33		.22
11			.48				.44
12				.62			.44
13				.77			.65
14	.77						.62
15						.38	.39
16	.58						.46
17			.39			.54	.48
18	.74						.61
19		.56					.34
20				.47			.34
21		.49					.40
22				.44			.38
23				.43			.49
24		.48					.31
25						.45	.24
26							.29
27					.78		.62
28							.21
29					.74		.59
30			.35				.42
% Total variance	8.28	7.73	7.47	7.18	5.74	5.60	
% Common variance	19.72	18.41	17.79	17.10	13.66	13.33	
Sum of communalities							12.60

Total variance accounted for 41.99

TABLE 6. PRINCIPAL AXIS FACTORING WITH DIAGONAL AS INPUT UNDER NORMAL VARIMAX ROTATION FOR TBMC GRADE EIGHT

Variable item	Factor						Communality h^2
	1	2	3	4	5	6	
1			.65				.50
2			.58				.35
3			.55				.35
4			.37	.33			.39
5			.51	.43			.48
6				.54			.38
7			.35				.32
8		.43					.38
9							.20
10						.68	.52
11						.69	.50
12				.64			.48
13				.46			.41
14	.73						.56
15	.37						.34
16	.64						.45
17	.72						.56
18	.73						.65
19		.52					.34
20		.57					.48
21		.64					.54
22		.35		.50			.43
23		.62					.49
24		.61					.45
25							.41
26	.49						.48
27					.70		.56
28						.56	.35
29					.69		.56
30	.42	.35		.37			.50
% Total variance	10.09	9.07	7.86	6.66	5.78	5.16	
% Common variance	22.61	20.33	17.62	14.92	12.95	11.57	
Sum of communalities							13.39

Total variance accounted for 44.62

At the grade six level, factor one has four items with loadings of .33 or greater. Item 8 is region comparison and 14, 16, and 18 involve region comparison with equal unit subdivision.

Factor two includes items 6 and 7, subdivision of a segment, and 19, 21, and 24, subdivision of a rectangle into five and six equal parts and a square into five parts.

Factor three includes items 1, 3, 4 and 5, all comparison of segmented paths, and 11 and 17, both area comparison, one comparison of elongated regions and the other having noncongruent units.

Factor four includes items 12 and 13, finding multiples of a given segment, and 20, 22 and 23, subdivision of a circle into five and six equal parts and a square into six parts.

Factor five includes item 10, a comparison of regions and items 27 and 28, comparisons of subdivision in regions with noncongruent subdivisions.

Factor six includes item 9, comparison of regions without subdivisions, items 15 and 17, area unit iteration involving noncongruent units, and item 25, comparison of congruent units from noncongruent regions. The factor structure for grade six is more complex and less interpretable than for grade four. It is summarized as follows:

1) area unit iteration involving congruent units,
2) linear subdivision including regions which can be divided unidimensionally,
3) linear unit iteration (operational linear measure),
4) multiplication of a segment and subdivision of regions, and
5) area unit iteration involving noncongruent regions and units.

At the grade eight level the six factors are quite well defined. Factor one has high loadings on items 14-18, area unit iteration, plus 26 and 30, comparing pairs of congruent regions which have been subdivided into an equal number of noncongruent units.

Factor two includes items 19-24, all subdivision of regions plus item 8, comparison of noncongruent regions which have no subdivisions.

Factor three consists of items 1-5, comparison of segmented paths, and item 7, subdivision of a segment into six parts.

Factor four includes items 4 and 5, segmented paths with all attributes varying, item 6, subdivision of a segment into five equal parts, items 12 and 13, constructing multiples of a segment, item 22, division of a square into six equal parts and item 30, comparison of noncongruent units from congruent regions.

Factor five involves items 27 and 29, comparison of noncongruent area units and factor six includes comparison of noncongruent regions and noncongruent subdivisions. Three very strong factors are apparent at this level with a fourth less interpretable factor as well. They are:

1) area unit iteration,
2) subdivision of regions,
3) linear unit iteration (operational linear measure),
4) [appears to be] synthesis operation involving unit size and number.

A survey of the factors from grades four to eight reveals some interesting trends. Items 19-24, subdivision of regions, have high loadings on four different factors at the grade four level, on two at grade six, and are nearly exclusive to one factor by grade eight.

The items involving subdivision of a segment and unit iteration given a unit segment are isolated on one factor in grade four but become synthesized with region subdivision in grade six. By grade eight, there seems to be a general assimilation of the more rigorous linear unit iteration items with area unit comparisons. At all levels the comparison of noncongruent regions having no unit subdivisions and comparison of noncongruent subdivisions of congruent figures proved very difficult. These items contributed very inconsistently to score variance except in the grade eight case where factor six has high loadings for items 10, 11, and 28. In view of the very rigorous reasoning necessary to complete the three items successfully, it is possible that not until grade eight are most students able to cope with these items. The factor structure which remains essentially the same across grade levels is:

1) an operational length measure factor,
2) an area unit iteration factor, and
3) an area subdivision factor.

These factors, and the more or less reasonable distribution of test items on other factors at each grade level, suggest the necessary construct validity for the measurement test.

Reliability refers to consistency throughout a series of measurements (Cronbach 1970). Measures of internal consistency were obtained through use of the Kuder-Richardson formula 20 which represents the average correlation obtained from all possible split-half reliability estimates, given items of unequal difficulty. Reliability coefficients for the measurement test at the three grade levels are presented in Table 7. If .65 is considered acceptable for group decisions (Mehrens

and Lehmann, 1973, p. 122), then the grades six and eight coefficients meet the necessary condition. The grade 4 estimate of .61 is low. The pilot study included a test-retest situation at the grade four level, however, which provided a coefficient of stability for the measurement test. The coefficient obtained at that time was .74, an acceptable level of reliability, since stability is the important form of reliability here.

TABLE 7. MEASUREMENT TEST RELIABILITY COEFFICIENTS AT THREE GRADE LEVELS USING THE K-R 20 FORMULA

Grade level	Reliability coefficient
4	.61
6	.70
8	.76

Of the three factors, length of test, group homogeneity and item difficulty, which could have affected the reliabilities reported here, two are of particular importance. In the sense that the test measures a developmental construct, scores are dependent upon age, and computation of a reliability coefficient at a particular age level necessarily reduces the estimate. Certain items were very difficult at all levels. Indeed, at the grade four level many items showed very low difficulty indices (see Appendix B for item analysis of measurement test). Lower

reliabilities are to be expected when a great deal of guessing has taken place. The reliability estimates reported here are considered acceptable for the present study.

The Fraction Achievement and Fraction Retention Tests

Content

The Fraction Achievement and Fraction Retention tests were designed to assess achievement of unit objectives as put forth in the school system mathematics program. Items for both tests were chosen from a pool selected from curriculum materials with approval of cooperating teachers. Although teachers did not know specifically which items were included, they were aware of the item type. Tables 8, 9 and 10 provide a list of the objectives tested and related items on the Fraction Achievement tests for each grade level. The Fraction Retention test is a parallel form of the Fraction Achievement test so objectives and items are similar to those listed in the tables.

Validity and Reliability of the Fraction Tests

Selection of items from the recommended program and subsequent agreement by teachers of suitability to objectives provides the needed evidence of *content validity*.

Reliability of the rational number tests is provided through two analyses. The KR-20 formula provides a measure of internal consistency, results of which are reported in Table 11. The coefficients range from .90 for grade eight retention to .96 for grade six retention. All estimates are well above the accepted level of .85 for making predictions concerning individuals (Mehrens and Lehmann, 1973, p. 123).

TABLE 8. FRACTION OBJECTIVES BEING TESTED AND RELATED ITEMS ON THE FRACTION ACHIEVEMENT TEST AT THE GRADE FOUR LEVEL

Description of objectives	Items
associating a fraction with the shaded/unshaded part of a region. Congruent subdivisions	2, 3, 4, 5, 8, 10, 11, 12, 14, 34, 35
as above with noncongruent subdivisions	1, 6, 7, 9, 13
locating points on the number line	36, 37, 38, 39
giving fraction name for points indicated on number line	40, 41, 42, 43
naming equivalent fractions given physical representations	10, 12, 13, 16, 18, 20
finding missing denominator in sets of equivalent fractions	21, 22, 23, 24, 25
generating sets of equivalent fractions	26, 27, 28, 29, 30, 31, 32, 33

TABLE 9. FRACTION OBJECTIVES BEING TESTED AND RELATED ITEMS ON THE FRACTION ACHIEVEMENT TEST AT THE GRADE SIX LEVEL

Description of objectives	Items
giving a fraction for the shaded/unshaded part of a region	1, 3, 4, 6, 7, 8, 10
same as above but with noncongruent parts	5, 11, 12
locating fraction on number line	17, 18, 19, 20, 21, 22
giving fraction name for point indicated on number line	23, 25, 27
giving equivalent fraction in physical representation	2, 9, 13
generating sets of equivalent fractions	29, 30, 31, 32, 33, 34
reducing to lowest terms	35, 36, 37, 38, 39
changing improper fraction to mixed number and mixed number to improper fraction	40, 41, 42, 43 44, 45, 46, 47
comparing fractions	48, 49, 50, 51, 52, 53
addition	54-46, 60-62, 66-69
subtraction	57-59, 63-65, 70-71
multiplication	72-77
division	78-83

TABLE 10. FRACTION OBJECTIVES BEING TESTED AND RELATED ITEMS ON THE FRACTION ACHIEVEMENT TEST AT THE GRADE EIGHT LEVEL

Description of objectives	Items
giving a fraction for the shaded/unshaded part of a region	1, 2, 5, 8, 9, 10
same as above but with non-congruent subdivisions	3, 4, 6, 7
locating fraction on number line	11, 12, 13, 14, 15, 16, 17
giving fraction naming indicated point on number line	18, 19, 20
reducing to lowest terms	21, 22, 23, 24
changing improper fractions to mixed numbers and mixed number to improper fraction	25, 26, 27 28, 29, 30
generating equivalent fractions	31, 32, 33, 34, 35, 36
comparing fractions	37, 38, 39, 40, 41
addition	42-49
subtraction	50-55
multiplication	56-61
division	62-68

TABLE 11. RELIABILITY ESTIMATES BASED ON THE K-R 20 FORMULA FOR RATIONAL NUMBER TESTS AT THREE GRADE LEVELS

Grade level	Achievement coefficient	Retention coefficient
4	.92	.92
6	.95	.96
8	.94	.90

TABLE 12. PPM COEFFICIENTS OF EQUIVALENCE FOR THE ACHIEVEMENT AND RETENTION TESTS AT THREE GRADE LEVELS

Grade level	Coefficient
4	.85
6	.91
8	.79

A further provision of reliability is essential for tests used to make inferences about knowledge in a content area. A *coefficient of equivalence*, computed as a Pearson Product Moment correlation coefficient between the two criterion measures at each grade level, is reported in Table 12. Coefficients of equivalence usually give low estimates of reliability in relation to other estimates such as internal consistency. The reliability estimates at the three grade levels are considered adequate for the purposes of this study.

The Observation Schedule

Construction

The present study seeks to establish the relationship between measurement concepts and rational number concepts as they are understood by students from different schools. The most obvious school attribute is residential location. Main effects of school on the criterion could be confounded by nuisance variables such as instructional variables which differ from school to school. Of particular interest to the present study is the instructional use of linear and area models in teaching rational number concepts. To provide insight into instructional variation with respect to model use, an observation schedule was developed. A copy of the observation schedule can be found in Appendix E. The purpose of this observation was simply to ascertain the presence of measurement models used in rational number instruction. The data collected are to be considered descriptive and noninferential, thus the analysis of development given below is at an adequate level.

The necessary schedule provides information on:

1) type of model and media used to present concepts,

2) discourse associated with the model,

3) portion of class time devoted to model use.

To suit the above needs, a category system was developed using multiple coding techniques on six dimensions. The dimensions are:

1) model — linear or area

2) media — e.g. blackboard, overhead projector, etc.

3) initiator — teacher or student

4) discourse — expository or inquiry

5) response representation — enactive, iconic, symbolic

6) response levels -- recall 1, 2, 3 synthesis

A unit is defined as any run of events involving the introduction and use of a particular model during verbal discourse. The observer writes the time at the beginning of the run, places a check in the appropriate category of each dimension and continues checking categories until the run is over. A run is over when:

1) a different model is introduced,

2) discussion leaves the model as a referent.

The observer notes the time at the end of each run. Figure 4 provides a sample of observer responses to the various dimensions. Notes are taken on a separate sheet to describe activities which are not model-related during the regular class time.

Time	Model		Media	Initiator		Discourse		Response representation			Response level			Time
	Area	Linear		Teacher	Student	Exp.	Inq.	E	I	S	1	2	3	
9:15	___	✓	BB	✓	___	✓	___	___	___	___	___	___	___	
	___	___	___	___	___	___	✓	___	___	✓	✓	___	___	9:16

Fig. 4. Observation schedule.

Reliability and Validity of the Observation Schedule

Reliability of the schedule depends upon inter-observer agreement and objectivity of coding. The researcher and an assistant collected all observation schedule data. A training period plus three school visits by both observers was sufficient for a high degree of observer agreement. In terms of the number of runs, there was exact agreement on types of models used and duration of the runs. Since the model categories are noninferential, the congruence of data is not surprising and expresses excellent reliability. The frequency of responses for expository and inquiry categories for a given run varied between observers although the configuration was similar. For example, one observer might show 12 inquiry tallies then two expository in a run whereas the other observer might show 10 inquiry and two expository. This type of discrepancy did not affect the data reduction method which involved interpretation of discourse into two components. discussion (question/answer) and lecture, or demonstration. The differences in number of tallies by observers therefore had no discernible effect on the results.

Validity is a measure of the degree to which the schedule provides information being sought. The content is appropriate, possibilities are exhaustive and categories are noninferential with respect to quantification of model use. Predictive validity, although desirable, cannot be provided for lack of a suitable criterion measure.

Considering the fairly gross method of data reduction necessary for this portion of the study, reliability and content validity are acceptable.

Summary

This chapter has provided a description of the development of data collection instruments, with special emphasis on item selection for the TBMC. Validity and reliability measures were reported for the three paper and pencil tests and found to be adequate for the purposes of this study. The observation schedule was also found to have acceptable levels of validity and reliability.

Chapter V

RESULTS OF THE DATA ANALYSIS

Introduction

Results of the data analysis are reported below in four main sections. The chapter has been divided into parts which coincide with the four questions forming the basis for the study. The questions were stated in Chapters I and II and their relationship to the purposes of the study discussed. Each question is restated below. The related hypotheses and results of statistical analyses are presented.

Results of the Data Analysis: Question I

Question I: *What is the nature of the relationship between measurement concepts and rational number concepts at the three grade levels?*

A principal components analysis was carried out on the TBMC, Fraction Achievement Test and Fraction Retention Test at each grade level. The purpose of this analysis was to provide a statistical basis for development of subtests of the three tests.

The TBMC was given to all students at the three levels. A principal factor analysis using ones in the correlation matrix diagonal was carried out at each grade level and six and ten factor solutions interpreted to provide subtest items. All factor solutions are the result of a single rotation using the normal varimax procedure.

The choice of subtests from the TBMC could have been made exclusively on the basis of objectives tested. One possibility would have been to include two subtests: one composed of all linear measure items

and the other composed of all area measure items. Another possibility would be to break the linear and area subtests into smaller subtests based on more explicit objectives. For example, the area measure subtest could be broken into four groups of items: region comparison with unit subdivisions; region comparison without unit subdivisions; subdivision of a region; and comparison of subdivisions of two regions. The linear subtest items could have been assigned to smaller subtests in a similar manner. Although such an approach is feasible, it is clear that many such possibilities exist. It is therefore prudent to add objectivity to the process through a data analytic approach. The factor analysis is an excellent tool for achieving the desired goal.

Factor solutions obtained under the analysis are composed of orthogonal components. Subtests based on those components are treated as independent variables in the canonical correlation analysis, a highly desirable condition which facilitates interpretation of results.

A principal components analysis was also carried out on the Fraction Achievement and Fraction Retention Tests at each grade level. Once again, factors provided a means of objectivity in item selection while insuring independence of the resulting subtests.

Assignment of items to subtests based on the factor analysis is given below for each grade level. With the conclusion of subtest development at each grade level the null hypothesis of independence is stated and the canonical correlation results presented.

Grade Four

a. Subtests of the TBMC

The strategy for assigning items to subtests consists of first extracting a small number (usually six) of factors and submitting them to a varimax rotation. On the basis of the rotated solution, items having factor loadings of .33 or more on a single factor are considered for inclusion in a subtest. The items are considered suitable if they test a single objective and do not have high factor loadings (greater than .33) on factors defining other objectives. Where an item loads .33 or greater on another subtest defining factor, it is considered for inclusion only if the factor pattern is otherwise consistent with other items in the subtest. At the grade four level this is accomplished through examination of both the six and five factor solutions. The solutions are presented in Tables 13 and 14.[1] Items selected are based generally on the six factor solution (see Table 14) as follows.

Factor one, including items 14 through 18 (see Appendix A for the TBMC items) with factor loadings ranging from .45 to .62, defines an *area measure* subtest. Factor two defines a *linear subdivision and simple unit iteration* subtest which includes items 6, 7, 12 and 13 with factor loadings ranging from .55 to .67. Factor four defines a *linear measure* subtest consisting of items 1 through 5. Factor loadings range from .33 to .65. Difficulty arises over factor three as a basis for

---

[1]High negative loadings are not included in the tables. Where a small number of factors were extracted, for example five and six factor solutions, high negative loadings were occasionally encountered. In every case, an extraction of a large number of factors, say fifteen, resulted in the item with high negative loading transferring to a high positive loading on a single unique factor. This situation did not affect the composition of subtests at any of the grade levels.

TABLE 13. PRINCIPAL FACTOR ANALYSIS OF TBMC (FIVE FACTOR SOLUTION)

GRADE FOUR

Items	Factors 1	2	3	4	5	Communality h^2
1				.55*		.34
2				.49*		.42
3				.59*		.40
4				.45*		.31
5				.56*		.36
6		.47*		.37*	.35*	.52
7		.66*				.52
8	.60					.41
9						.11
10						.07
11						.25
12					.73*	.64
13					.57*	.50
14	.54*	.35*				.42
15	.56*					.39
16	.59*					.37
17	.62*					.44
18	.61*					.45
19						.25
20			.46*			.34
21						.33
22			.59*		.50*	.60
23			.47*			.28
24			.36*			.29
25						.29
26			.54			.37
27						.45
28		.35				.14
29						.32
30	.34	.38				.40
% Total variance	9.39	7.56	6.94	6.65	6.04	
% Common variance	25.67	20.67	18.96	18.18	16.51	
Sum of communalities						10.93

Total variance accounted for 36.58

*Items included on subtests.

TABLE 14. PRINCIPAL FACTOR ANALYSIS OF TBMC (SIX FACTOR SOLUTION)

GRADE FOUR

Items	Factors						Communality h^2
	1	2	3	4	5	6	
1				.45*		.71	.38
2							.44
3				.65*			.46
4				.48*			.32
5				.55			.38
6		.67*					.54
7		.67*					.52
8	.58						.42
9							.27
10							.09
11							.33
12		.55*			.53*		.68
13		.57*					.51
14	.45*		.33*				.48
15	.58*						.39
16	.62*						.47
17	.62*						.53
18	.53*						.48
19							.27
20			.52				.43
21				.44			.34
22			.44*		.48*	.40*	.63
23			.49*				.31
24							.29
25							.54
26			.53				.40
27					.34		.47
28							.38
29							.35
30			.50				.40
% Total variance	8.65	8.11	7.61	7.68	5.34	5.23	
% Common variance	20.78	19.49	18.29	16.05	12.83	12.57	
Sum of communalities							12.48

Total variance accounted for 41.60

*Items included on subtests.

a final subtest. All area subdivision items load positively on factor three with items 20, 22 and 23 having loadings ranging from .44 to .52. Inspection of the last two factors, five and six, indicates that item 22 also has a loading in common with two different linear items. This suggests a linear component to item 22. Inspection of the five factor solution (see Table 13) shows similar factor structure. Again item 22 shows a linear component on factor five. The decision is to include items 20, 22, 23 and 24 in an *area subdivision* subtest with recognition that it is not strictly independent of the others.

Extraction of a larger number of factors can change the clustering of item loadings on factors, thereby affecting subtest composition. It is desirable to show that the subtests remain interpretable with extraction of a greater number of factors. The ten factor solution is presented in Appendix C for this purpose. Factor one shows high loadings for area measure items 14, 16, 17, 18 with a loading of .28 for item 15 which was chosen as part of the *area measure* subtest. Tracing the pattern for item 15 shows two high loadings, one on factor two, a linear factor, and one on factor eight, an area factor. The *area measure* subtest is retained since item 15 has higher loadings on area components than on linear components.

Factor two shows high loadings for all items of the *linear subdivision and simple unit iteration* subtest. The *area measure* item 15 mentioned previously, and item 22 from the *area subdivision* subtest, have loadings of .48 and .35, respectively, on factor two. Item 7 shows a loading of .42 on factor three with items 21 and 24, both area subdivision items which can be subdivided along one dimension.

The subtest composed of items 6, 7, 12 and 13 is related to both of the area subtests. Factors four and six show loadings greater than .33 for items associated with the *area subdivision* subtest. Factors five and seven show high loadings for items of the *linear measure* subtest. It is accepted as independent of other subtests.

Four subtests have been selected on the basis of factor interpretation. The subtests are not strictly independent measures but were chosen for the closest approximation available while maintaining enough items to justify subtest size. A summary of subtests and items is given below.

Subtest 1. Linear Measure — items 1, 2, 3, 4, 5.

Subtest 2. Linear Subdivision — 6, 7, 12, 13
Simple Unit Iteration.

Subtest 3. Area Measure — 14, 15, 16, 17, 18.

Subtest 4. Area Subdivision — 20, 22, 23, 24.

b. Subtests of the Fraction Achievement Test

Unlike the TBMC, achievement and retention subtests are easily established on the basis of objectives being tested. Once again, it is desirable to have independence among subtests and the principal factor analysis provides a basis for item selection given that condition. Development of subtests begins with inspection of the six factor solution (Table 15).

Items 2, 4, 11, 15, 17 and 19 have factor loadings greater than .33 on factor one. All items involve naming a fractional part of a region with noncongruent subdivisions. Items 10, 12, 14, 16, 18 and 20 also have loadings greater than .33 on factor one. The items require naming an equivalent fraction given the shaded portion. Items 36, 38,

TABLE 15. PRINCIPAL FACTOR ANALYSIS OF FRACTION ACHIEVEMENT TEST (SIX FACTOR SOLUTION) GRADE FOUR

Items	Factors						Communality h^2
	1	2	3	4	5	6	
1				.66*	.48*		.76
2	.86*						.80
3				.48*			.47
4	.77*						.63
5				.70			.58
6				.71*			.59
7				.76*	.35*		.79
8				.57	.61*		.70
9					.73*		.64
10	.53		.37				.53
11	.77*						.63
12	.60						.44
13					.63*		.48
14	.40		.42				.34
15	.84*						.77
16	.62		.41				.65
17	.81*						.70
18	.73						.59
19	.62*		.50*				.64
20	.62		.49				.63
21		.45*					.32
22		.42*			.40*		.43
23		.61*					.60
24		.38*			.41*	.36*	.55
25		.46*				.44*	.53
26		.72*					.57
27		.81*					.69
28		.74*					.58
29		.75*					.64
30		.58*				.43*	.63
31		.54*				.39*	.56
32						.80*	.74
33						.80*	.75
34			.65				.58
35			.66				.63
36	.57*			.37*			.50

(Cont'd)

TABLE 15 (Cont'd)

Items	Factors 1	2	3	4	5	6	Communality h^2
37							.23
38	.44*			.37*			.49
39			.77*				.63
40	.59*						.43
41	.53*						.40
42			.72*				.59
43			.77*				.67
% Total variance	17.79	10.83	9.18	7.94	6.47	6.12	
% Common variance	30.51	18.56	15.73	13.61	11.10	10.49	
Sum of communalities							25.08

Total variance accounted for 58.32

*Items included on subtests.

40 and 41 also have high loadings on factor one. All items involve placing fractions on the number line.

Factor three has moderate loadings for items 10, 14, 16, 19 and 20 ranging from .37 to .50. All require giving an equivalent fraction for a shaded part of a region. The items also have high loadings on factor one (.40 to .62). Items 39, 42 and 43 have high loadings (.72 to .77) on factor three, locating fractions on the number line. The two factors, one and three, appear to represent an initial fraction concept involving both linear and area attributes but the region items represent a higher level of difficulty. In deciding upon subtests from the two factors, the decision is made to choose region items which show the least relationship to number line items for one subtest. Items 2, 4, 11, 15, 17 and 19 are combined to form an *initial fraction* subtest involving *noncongruent* subdivisions. Items 36, 37, 38, 39, 40, 41, 42 and 43 are combined to form a *number line* subtest. In view of the high loadings on factor one for number line items 36, 38, 40 and 41, the two subtests are related.

Factors two and six have high loadings for items 21 to 31 and items 24, 25, 30, 31, 32 and 33. The items test equivalence in a symbolic context and are combined to form an *equivalence* subtest. Two of the items have moderate loadings on factor five but the relationship between the *equivalence* subtest and other subtests is considered weak.

Initial fraction items 1, 3, 6, 7, 9 and 13 have high loadings on factors four and five. Two *number line* items and two *equivalence* items have loadings greater than .33 on factors four and five respectively.

The initial fraction items combine to form an *initial fraction* subtest involving *congruent* subdivision. The subtest is related to *number line* and *equivalence*.

The subtests defined on the Fraction Achievement Test (see Appendix A for fraction test items) six factor solution are:

Subtest 1. Initial Fraction — items 1, 3, 6, 7, 9, 13. (congruent units)

Subtest 2. Initial Fractions — 2, 4, 11, 15, 17, 19. (noncongruent units)

Subtest 3. Equivalence — 21, 22, 23, 24, 25, 26, 27, 29, 29, 30, 31, 32, 33.

Subtest 4. Number Line — 36, 37, 38, 39, 40, 41, 42, 43.

None of the subtests is completely independent, although subtest 3 shows only a weak relationship to subtest 1. The strongest relationship exists between subtests 2 and 4.

Inspection of the ten factor solution (see Appendix C) shows an expansion of the existing factors with items from one factor moving to two or more. No further relational difficulties are noted.

c. Subtests of the Fraction Retention Test

A principal factor analysis was carried out on the retention test (see Appendix A for fraction test items). Table 16 shows the five factor solution.

Factor one has items 2, 3, 4, 11, 17 and 19 with loadings greater than .33. All require giving the fractional names for a shaded portion of a region which has noncongruent subdivisions. The items form the *initial fraction* subtest with noncongruent units.

Factor three has high loadings for items 21, 22, 23, 26, 27, 28, 29, 30 and 31, all equivalence items. Items 24 and 25, also equivalence

TABLE 16. PRINCIPAL FACTOR ANALYSIS OF FRACTION RETENTION TEST (FIVE FACTOR SOLUTION) GRADE FOUR

Items	Factors					Communality h^2
	1	2	3	4	5	
1		.58*				.46
2	.79*					.73
3	.83*					.77
4	.81*					.71
5		.75*				.63
6		.76*				.68
7		.72*				.63
8		.73				.61
9	.35	.67				.61
10	.54			.41		.54
11	.75*					.60
12	.67					.59
13		.75*				.68
14	.58			.39		.59
15		.69*				.54
16		.34				.38
17	.76*					.59
18	.66					.55
19	.71*					.60
20	.63					.53
21			.58*			.43
22			.43*		.66*	.66
23			.54*			.46
24					.75*	.67
25					.81*	.76
26			.70*			.52
27			.75*			.58
28			.79*			.69
29			.79*			.69
30			.71*			.64
31			.62*		.33*	.59
32				.65		.52
33			.34	.65		.56
34				.59		.39
35				.58		.43
36	.34			.42*		.33

(Cont'd)

TABLE 16 (Cont'd)

Items	Factors					Communality
	1	2	3	4	5	h^2
37				.40*	.42*	.35
38					.34*	.27
39				.36*		.25
40				.50*		.33
41				.39*		.28
42				.55*		.31
43						.11
% Total variance	15.51	11.45	10.80	8.60	6.72	
% Common variance	29.22	21.57	2 .36	16.19	12.66	
Sum of communalities						22.82

Total variance accounted for 53.08

*Items included on subtests.

items, both have factor loadings of .31, very close to the criterion of .33. Items 21 to 31 inclusive form the *equivalence* subtest.

Factor four has high loadings for items 36, 37, 39, 40, 41 and 42, all number line items. The other four items loading at .33 or greater are initial fraction items, none of which are included in subtests. Items 36 to 43 are combined to form the *number line* subtest.

Factor five has three *equivalence* items with high loadings and two *number line* items with high loadings. This factor indicates a relationship exists between the *equivalence* and *number line* subtests.

The four subtests which emerge from the five factor solution are listed below.

Subtest 1. Initial Fraction — items 1, 6, 7, 13, 15. (congruent units)

Subtest 2. Initial Fractions — 2, 3, 4, 11, 17, 19. (noncongruent units)

Subtest 3. Equivalence — 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31.

Subtest 4. Number Line — 36, 37, 38, 39, 40, 41, 42, 43.

Inspection of the ten factor solution (see Appendix C) indicates a splitting of the five factors to a supporting configuration. Factors one and two have the same items with even higher loadings, ranging from .71 to .83 on factor one (subtest 2) and .46 to .79 on factor two (subtest 1). Factors three and four together have items 21 to 31 with high loadings on either factor or both factors. The items form the *equivalence* subtest, which appears to be independent of other subtests under this solution.

Factor five has items from the *number line* subtest and an *initial fraction* item with loadings of .33 or greater. This supports

the relationship between the subtests which was established on the five factor solution.

Factors eight, nine and ten have all number line items with high loadings.

The factor structure is consistent from the five to ten factor solutions.

The factor structure is similar for the Fraction Achievement and Fraction Retention tests. The selected subtests include comparable items from the two tests so that comparisons between subtests are valid.

d. Canonical Correlation Analyses for Grade Four

The relationship between the four TBMC subtests and the four fraction subtests is established through two canonical correlation analyses. The canonical correlation analysis establishes the interrelationships between the two sets of measurements, TBMC subtest scores and fraction subtest scores, made on the same students at each grade level. By determining the linear function, on the TBMC subtests, that has maximum covariance with a linear function on the fraction subtests, the first canonical correlation is established. The two linear functions are maximally correlated. Other linear functions with maximum correlation are also determined with the restriction that the new functions have zero correlation with the previous functions. The derived linear functions, or canonical variates, are treated as regression weights for the original variables. In the case of the grade four analyses, four subtests are included on either side of the equation. The number of different sources of variance possible between the two sets of subtests is four. The significance of the four correlations

between the pairs of linear functions is tested by the Bartlett Lambda procedure (Cooley and Lohnes 1968). Only those weighting systems associated with significant canonical correlations are reported as normalized weights. The first analysis involves the TBMC subtests as predictors and the Fraction Achievement subtests as criterion variables. In the second analysis the TBMC subtests again are predictors but criterion variables are scores from the Fraction Retention subtests. The hypothesis being tested is:

> Hypothesis I. *TBMC subtest scores are independent of fraction subtest scores at the grade four level*
>
> *a) for the Fraction Achievement subtests as criterion variables,*
>
> *b) for the Fraction Retention subtests as criterion variables.*

(i) Fraction Achievement Subtests as criterion variables. The matrix of correlations among the predictor and criterion subtests is presented in Appendix D. Results of the canonical correlation derived from the correlation matrix are reported in Table 17. Information given includes a list of the correlations associated with each weighting system, the Chi-square value for testing each correlation, and a list of the weights associated with each significant correlation.

Using the achievement subtests as criterion, one significant canonical correlation emerges. The two sets of subtests are related and the hypothesis of independence is rejected. The weights associated with the significant correlation indicate which subtests contribute most to the relationship. It can be seen from Table 17 that achievement subtests 1, 2 and 3 and TBMC subtests 2 and 3 contribute most heavily. The strongest relationship exists between achievement subtest 2

TABLE 17. CANONICAL CORRELATION ANALYSIS OF THE TBMC SUBTESTS AND FRACTION ACHIEVEMENT SUBTESTS FOR GRADE FOUR

Canonical Correlation	Chi Square	df	P
.48	34.56	16	.005
.19	6.65	9	.673
.15	3.01	4	.556
.08	.74	1	.391

Weighting System Associated with Significant Canonical Correlation

TBMC subtest	*Weight*	*Achievement subtest*	*Weight*
1. Linear measure	.19	1. Initial fractions (congruent units)	.50
2. Linear subdivision simple unit iteration	.75	2. Initial fractions (noncongruent units)	.72
3. Area measure	.60	3. Equivalence	.48
4. Area subdivision	.21	4. Number line	-.02

with a weight of .72 and TBMC subtest 2 with a weight of .75. The number line subtest has no effect on the relationship and the practical effects of Linear measure and Area subdivision are considered negligible.

(ii) Fraction Retention Subtests as criterion variables. Results of the canonical correlation analysis are reported in Table 18. The first correlation is significant. A relationship is established and the hypothesis of independence is rejected. The weights associated with the significant canonical correlation are listed in Table 18. The *initial fraction* subtest 2 has a weight of .82, indicating it makes the greatest contribution to the relationship. *Initial fraction* subtest 1 also contributes to the relationship in a practical sense. Of the TBMC subtests, 2 and 3 are again the most important. Subtest 1 contributes very little with a weight of .26. All subtests make a measurable contribution with strongest relationship between the *linear subdivision* and *area measure* subtests with the two *initial fraction* subtests.

Grade Six

a. Subtests of the TBMC

The process followed for extraction of subtest items is similar to that outlined for the grade four test. Subtest composition is determined on the basis of the six factor solution shown in Table 19.

Factors one and six have high loadings for eight area items. Of the eight, items 14, 16 and 18 load on factor one and items 15 and 17 on factor six. Taken together, items 14 through 17 form the *area measure* subtest.

Factor two has high loadings for both linear and area subdivision and factor four includes simple linear unit iteration items plus

TABLE 18. CANONICAL CORRELATION ANALYSIS OF TBMC SUBTESTS AND FRACTION RETENTION SUBTESTS FOR GRADE FOUR

Canonical Correlation	Chi Square	df	P
.59	53.20	16	.000
.21	8.24	9	.510
.16	3.33	4	.504
.08	.60	1	.440

Weighting System Associated with the Significant Canonical Correlation

TBMC subtests	*Weight*	*Fraction retention subtests*	*Weight*
1. Linear measure	.26	1. Initial fractions (congruent units)	.46
2. Linear subdivision simple unit iteration	.69	2. Initial fractions (noncongruent units)	.82
3. Area measure	.57	3. Equivalence	.21
4. Area subdivision	.35	4. Number line	.27

TABLE 19. PRINCIPAL FACTOR ANALYSIS OF TBMC (SIX FACTOR SOLUTION) FOR GRADE SIX

Items	Factors 1	2	3	4	5	6	Communality h^2
1			.62*				.47
2							.18
3			.60*				.37
4			.39*				.36
5			.50*				.52
6		.64*					.53
7		.71*					.56
8	.61						.44
9						.41	.25
10					.33		.22
11			.48				.44
12				.62*			.44
13				.77*			.65
14	.77*						.62
15						.38*	.39
16	.58*						.46
17			.39*			.54*	.48
18	.74*						.61
19		.56*					.34
20				.47*			.34
21		.49*					.40
22				.44*			.31
23				.43*			.49
24		.48*					.31
25						.45	.24
26							.29
27					.78		.62
28							.21
29					.74		.59
30			.35				.42
% Total variance	8.28	7.73	7.47	7.18	5.74	5.60	
% Common variance	19.72	18.41	17.79	17.10	13.66	13.33	
Sum of communalities							12.60

Total variance accounted for 41.99

*Items included on subtests.

subdivision of regions. The factors reflect skills common to both the linear and area items. Items 6, 7, 12, 13 and 19 through 24 are combined to form the *linear and area subdivision subtest.*

Factor three has four linear items, 1, 3, 4 and 5 with moderate to high loadings and one area item, 17, selected for the *area measure* subtest. Item 2, one of the preconceived linear measure items, has a loading of .26. Items 1 through 5 are combined to form the *linear measure* subtest. The three subtests formed from the TBMC are:

Subtest 1. Linear Measure — items 1, 2, 3, 4, 5.
Subtest 2. Area Measure — 14, 15, 16, 17, 18.
Subtest 3. Linear and Area Subdivision — 6, 7, 12, 13, 19, 20, 21, 22, 23, 24.

The appearance of a moderate loading for item 17 on factor three indicates a small relationship between subtests 1 and 2.

Inspection of the ten factor solution (see Appendix C), indicates that relationships exist among all three subtests. Factor one remains relatively unchanged from the five factor solution. Items 15 and 17 have high loadings on factors two, four and nine, which also include subdivision and a linear item with high loadings.

Factor three includes the four linear measure items included in the six factor solution but also has a high loading for item 6, linear subdivision.

Factors two, four and six seem to be an expansion of factors two and four from the five factor solution but now include an area measure and a linear measure item. The *linear and area subdivision* subtest is not independent of the other two subtests. The three subtests are retained as defined on the six factor solution although at this grade

level the factors, hence the subtests, include a synthesis of linear and area items.

b. Subtests of the Fraction Achievement Test

The Fraction Achievement Test contains 83 items at the grade six level. Initial subtest construction is on the basis of a nine factor solution as shown in Table 20.

Items 54 through 71, all addition or subtraction, have high loadings on factor one. Four other item types have moderate loadings on factor one. Items 54 through 71 are combined to form an *addition/subtraction* subtest.

Factor two has loadings of .49 to .77 for items 18 to 24, 27 and 29, all number line items. The items are combined to form a *number line* subtest.

Factor three has loadings of .45 to .75 for items 40 through 47, items requiring changes from mixed fraction form to improper fraction form. The items are combined to form a *renaming* subtest. Other moderate loadings on factor three range from .34 to .49. These items involve four different fraction objectives.

Factor four has items 2, 3, 5 and 7 through 13 with loadings of .34 to .80. The items form an *initial fraction* subtest including regions with both congruent and noncongruent subdivisions.

Factor five has items 78 to 83 with loadings .46 to .67, all division items. Two multiplication items have loadings of .43 and .50 on factor five. Two items from *addition/subtraction* and one from *initial fractions* also have low loadings on factor five. Items 78 to 83 are combined to form a *division* subtest.

TABLE 20. PRINCIPAL FACTOR ANALYSIS OF THE FRACTION ACHIEVEMENT TEST (NINE FACTOR SOLUTION) GRADE SIX LEVEL

Items	Factor									communality
	1	2	3	4	5	6	7	8	9	h^2
1								.61*		.51
2				.34*				.51*		.51
3				.35*				.54*		.44
4										.26
5				.68*						.57
6									.55	.38
7										.27
8				.55*						.44
9				.55*						.38
10				.57*	.33*					.54
11				.80*						.70
12				.80*						.67
13				.68*						.55
14									.81	.68
15									.79	.67
16									.50	.37
17										.30
18										.37
19		.75*								.66
20		.73*								.62
21		.76*								.63
22		.49*								.37
23		.77*								.65
24		.76*								.64
25			.35							.30
26										.57
27		.77*								.66
28		.76*								.66
29			.47*							.50
30						.41*				.44
31				.46*		.41*				.47
32	.45*									.51
33	.38*									.45
34	.35*									.45
35			.36							.56
36			.49							.43
37	.46									.45
38	.47									.47
39									.33	.35
40			.75*							.67

(Cont'd)

TABLE 20 (Cont'd)

Items	Factor 1	2	3	4	5	6	7	8	9	commu-nality h^2
41			.61*							.59
42			.65*							.49
43			.71*							.59
44			.54*							.48
45			.45*							.30
46	.36*		.53*							.59
47			.59*							.56
48						.72*				.66
49						.74*				.69
50						.73*				.65
51	.36*					.73*				.31
52						.68*				.51
53						.41*				.38
54	.60*									.53
55	.77*									.66
56	.73*									.60
57	.58*		.44*							.56
58	.66*		.34*							.61
59	.71*									.60
60	.60*				.44*					.64
61	.77*									.82
62	.73*									.72
63	.63*									.50
64	.68*					.34*				.69
65	.68*									.67
66	.70*				.33*					.73
67	.62*									.64
68	.64*							.39*		.71
69	.63*							.40*		.69
70								.39*		.30
71	.43*							.62*		.64
72							.78*			.68
73							.85*			.77
74							.76*			.70
75					.50*		.36*			.51
76					.43*					.42
77							.65*			.55
78					.46*					.40
79					.59*					.45
80					.61*					.51

(Cont'd)

TABLE 20 (Cont'd)

Items	Factor 1	2	3	4	5	6	7	8	9	commu-nality h^2
81					.58*					.44
82					.63*					.65
83					.67*					.52
% Total variance	12.17	6.71	6.65	6.03	5.77	4.89	4.22	4.08	3.76	
% Common variance	22.43	12.36	12.25	11.16	10.63	9.00	7.77	7.52	6.93	
Sum of communalities										45.05

Total variance accounted for 54.27

*Items included on subtests.

Factor six has high loadings for items 48, 49, 50, 52 and 53, all fraction comparisons. They are combined to form a *comparing* subtest.

Factor seven shows loadings of .36 to .85 for items 72, 73, 74, 75 and 77. Item 76 has a loading of .30, slightly less than criterion. Items 72 to 77 are combined to form a *multiplication* subtest.

Items 29 through 34 show low but similar loadings on three factors. (See Appendix C for the nine factor solution matrix.) On factor one the loadings range from .21 to .45, on factor four from .29 to .46, and on factor six from .15 to .41. Loadings on other factors generally are consistent for these items. A decision was made to include items 29 through 34 on an *equivalence* subtest since the items are testing the same objective and factor patterns are similar. The subtest is related to the *addition/subtraction*, *initial fractions*, and *comparing* subtests.

The eight subtests which emerged from the analysis are:

Subtest 1. Initial Fractions — items 1, 2, 3, 5, 8, 9, 10, 11, 12, 13.

Subtest 2. Number Line — 19, 20, 21, 22, 23, 24, 27, 28.

Subtest 3. Equivalence — 29, 30, 31, 32, 33, 34.

Subtest 4. Renaming — 40, 41, 42, 43, 44, 45, 46, 47.

Subtest 5. Comparing — 48, 49, 50, 51, 52, 53.

Subtest 6. Addition/Subtraction — 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71.

Subtest 7. Multiplication — 72, 73, 74, 75, 76, 77.

Subtest 8. Division — 78, 79, 80, 81, 82, 83.

Examination of the fifteen factor solutions (see Appendix C), shows fewer relationships among subtests. Factor one with factors nine and fifteen have high loadings nearly exclusive to subtest six.

Factor two has high loadings only on the number line items of subtest two.

Factor three has high loadings for all the renaming items, those of subtest four, plus two items which were not included in a subtest.

Factor four with factor thirteen include high loadings for the *initial fraction* items, subtest one, with a low loading for one subtraction item on factor thirteen.

Factor five has high loadings for the *multiplication* subtest items.

The high loading *comparing* items are exclusive to factor six, *division* items load exclusively on factor seven, and *equivalence* items load exclusively on factor eight.

The fifteen factor solution supports the choice of subtests and indicates one very weak relationship among the *initial fraction* subtest and the *addition/subtraction* subtest.

c. Subtests of the Fraction Retention Test

Selection of subtest items was based on the nine factor solution of the principal factor analysis which is presented in Table 21. Factor one has moderate to low loadings for several items but moderate to high loadings, from .54 to .80, for items 58 to 75. The items form the *addition/subtraction* subtest.

Factor two has high loadings for items 21 through 32. The items are combined to form the *number line* subtest.

Factor three has two sets of items with moderate to high loadings. Items 33 to 38 with loadings of .40 to .80 are selected to form an *equivalence* subtest. Items 48 to 51, also with loadings greater than

TABLE 21. PRINCIPAL FACTOR ANALYSIS OF THE FRACTION RETENTION TEST (NINE FACTOR SOLUTION) GRADE SIX LEVEL

Items	Factor 1	2	3	4	5	6	7	8	9	communality h^2
1									.63*	.47
2					.41	.38			.39	.62
3			.43*							.37
4										.27
5										.27
6										.19
7	.42									.39
8				.45*						.39
9			.40		.43					.49
10										.15
11					.64*	.40*				.64
12					.63*					.62
13					.75					.63
14					.48*					.40
15			.47		.47					.57
16					.64*					.55
17					.54			.35		.53
18									.54	.53
19				.39					.39	.54
20									.45	.26
21	.35*	.36*								.51
22		.35*			.47*					.52
23		.83 *								.77
24		.70*								.59
25		.80*								.75
26		.42*			.37*					.51
27		.82*								.70
28		.80*								.70
29		.86*								.79
30		.86*								.79
31		.82*								.73
32		.86*								.77
33			.80*							.80
34			.65*							.61
35			.40*							.34
36	.38*		.70*							.68
37			.67*							.63
38			.43*			.41*				.52
39	.46					.43				.51
40	.37					.40				.46

(Cont'd)

TABLE 21 (Cont'd)

Items	Factor 1	2	3	4	5	6	7	8	9	commu-nality h^2
41	.39					.38				.45
42	.45									.39
43	.38									.45
44						.74*				.62
45						.64*				.64
46						.64*				.54
47						.67*				.62
48			.63*			.36*				.66
49	.36*		.39*							.44
50	.35*		.45*							.49
51			.65*			.37*				.72
52								.66*		.67
53								.74*		.68
54								.65*		.58
55	.41*			.39*				.34*		.54
56								.64*		.51
57										.32
58	.64*								.42*	.66
59	.58*								.76*	.62
60	.55*								.47*	.56
61	.75*									.74
62	.68*									.64
63	.59*									.53
64	.74*									.67
65	.54*									.43
66	.70*									.56
67	.54*									.39
68	.69*									.67
69	.68									.64
70	.80*									.74
71	.70*									.64
72	.73*									.64
73	.72*									.68
74	.55*									.42
75	.78*									.71
76				.88*						.84
77				.83*						.75
78				.69*						.65
79				.55			.49*			.61
80				.34			.50*			.51
81				.72*						.63
82				.55*						.52
83							.72*			.63

(Cont'd)

TABLE 21 (Cont'd)

Items	Factor 1	2	3	4	5	6	7	8	9	Communality h^2
84							.74*			.71
85							.69*			.60
86							.54*			.42
87							.71*			.65
% Total variance	13.56	8.64	6.38	5.60	5.23	4.89	4.82	3.96	3.58	
% Common variance	23.94	15.25	11.26	9.88	9.23	8.62	8.51	6.98	6.32	
Sum of communalities										49.30

Total variance accounted for 56.66

*Items included on subtests.

.33, indicate the existence of a relationship between *equivalence* and *renaming* subtest.

Factor four has one group of items, 76 to 81, all *multiplication*, loading .34 to .88 to form a subtest.

Factor seven has moderate loadings for items 79 and 80, .49 and .50 respectively, both multiplication items and high loadings, .54 to .74 for items 83 to 87. The last group are combined to form the *division* subtest. A relationship clearly exists between the multiplication and division subtests.

Factor eight with items 52 to 56 having loadings from .34 to .74 define the *comparing* subtest.

The eight subtests extracted from the nine factor solution are:

Subtest 1. Initial Fractions — items 1, 3, 5, 8, 10-12, 14, 16.
Subtest 2. Number Line — 21-32.
Subtest 3. Equivalence — 33-38.
Subtest 4. Renaming — 44-51.
Subtest 5. Comparing — 52-57.
Subtest 6. Addition/Subtraction — 58-75.
Subtest 7. Multiplication — 76-81.
Subtest 8. Division — 82-87.

Relationships exist between subtests 3 and 4 and subtests 7 and 8.

Inspection of the fifteen factor solution (see Appendix C) shows factors one and three can be combined to produce subtest 6. Factor two supports the choice of items for subtest 2 and factor four supports the choice of items for subtest 7. The *equivalence* subtest items are represented on factor five along with a renaming item indicating that relationship is maintained.

Factor six supports the choice of items for subtest 8 along with the existence of a relationship with subtest 7.

Factors eight and ten combine to give subtest 4, and factor nine supports the choice of items for subtest 5.

Factor seven with eleven and fifteen provide supporting data for items selected to comprise subtest 1. The subtests and relationships established through the nine factor solution are supported by the fifteen factor solution.

d. Canonical Correlation Analyses for Grade Six

The three TBMC subtests and each set of fraction test subtests were submitted to a canonical correlation analysis using the TBMC subtests as predictors and first the achievement subtests as criterion variables and then the retention subtests.

The hypothesis being tested is:

Hypothesis I. *TBMC subtest scores are independent of fractional subtest scores at the grade six level*

a) for the Fraction Achievement subtests as criterion variables,

b) for the Fraction Retention subtests as criterion variables.

(i) Fraction Achievement Subtests as criterion variables. The matrix of correlations among the three predictor and eight criterion variables is presented in Appendix D. Results of the canonical correlation analysis derived from the correlation matrix are reported in Table 22. The three canonical correlations associated with each weighting system and the Chi-square value for testing each correlation are listed.

TABLE 22. CANONICAL CORRELATION ANALYSIS OF TBMC SUBTESTS AND FRACTION ACHIEVEMENT SUBTESTS GRADE SIX

Canonical Correlation	Chi Square	df	P
.53	60.81	24	.000
.33	18.04	14	.205
.16	3.39	6	.759

System of Weights Associated with the Significant Canonical Correlation

TBMC subtest	*Weight*	*Achievement subtest*	*Weight*
1. Linear measure	.45	1. Initial fractions	.66
2. Area measure	.43	2. Number line	.09
3. Linear and area subdivision	.79	3. Equivalence	.28
		4. Renaming	.15
		5. Comparing	.45
		6. Addition/Subtraction	.49
		7. Multiplication	.07
		8. Division	.09

The first canonical correlation of .53 is the only statistically significant value. The null hypothesis of independence is rejected. The system of weights associated with the significant canonical correlation is listed at the bottom of Table 22. The strongest relationship, as evidenced by subtest weights, is between TBMC subtest 3 and achievement subtest 1. Also of practical importance are TBMC subtests 1 and 2 and achievement subtests 5 and 6. Subtests which contribute essentially nothing to the relationship are achievement subtests 2, 4, 7 and 8.

(ii) Fraction Retention subtests as criterion variables. The correlation matrix for the three TBMC and eight retention subtests is reported in Appendix D. Results of the canonical correlation analysis are listed in Table 23. The first canonical correlation is significant so the null hypothesis is rejected. Inspection of the weighting system associated with the significant correlation indicates which variables (subtest scores) contribute most to the relationship.

The three TBMC subtests contribute nearly equal amounts to the relationship. Of the retention subtests, subtests 1, 3 and 5 make similar contributions and subtest 4, with a negative weight, has a negative relationship to the other subtests. Subtests 2, 7 and 8 of the retention subtests make no practical contribution to the relationship.

Grade Eight

a. Subtests of the TBMC

Subtests were constructed on the basis of the principal factor analysis (six factor solution) of the TBMC and checked for validity

TABLE 23. CANONICAL CORRELATION ANALYSIS OF TBMC SUBTESTS AND FRACTION RETENTION SUBTESTS GRADE SIX

Canonical Correlation	Chi Square	df	P
.59	68.82	24	.000
.27	13.71	14	.472
.17	3.84	6	.699

System of Weights Associated with the Significant Canonical Correlation

TBMC subtest	*Weight*	*Retention subtest*	*Weight*
1. Linear measure	.61	1. Initial fractions	.51
2. Area measure	.56	2. Number line	-.05
3. Linear and area subdivision	.57	3. Equivalence	.49
		4. Renaming	-.36
		5. Comparing	.52
		6. Addition/Subtraction	.28
		7. Multiplication	-.04
		8. Division	.14

with the ten factor solution. The six factor solution is presented in Table 24.

Three factors are distinguished by the presence of sequences of high loading items (see Appendix A for TBMC items). Factor one has five items with loadings of .37 to .73, all area measure items. Factor two has loadings of .35 to .64 for items 19 through 24, area subdivision items, and factor three has six linear items with loadings of .35 to .65. Together, factor three and four provide the *linear* subtest items.

The three subtests which result from the six factor solution are:

Subtest 1. Linear measure — items 1, 2, 3, 4, 5, 7, 13.
Subtest 2. Area measure — 14, 15, 16, 17, 18, 26, 30.
Subtest 3. Area subdivision — 19, 20, 21, 22, 23, 24.

An examination of the ten factor solution (see Appendix C), indicates a separation of linear measure items to three factors. A relationship is apparent between area subdivision items and linear measure items on factor four. In general, the choice of subtests is supported.

b. Subtests of the Fraction Achievement Tests

Subtests were constructed on the basis of an eight factor solution as shown in Table 25. The first factor has high loadings for all multiplication and division items, 56 through 68, with loadings ranging from .44 to .73. Items 56 through 68 are combined to form a *multiplication/division* subtest.

TABLE 24. PRINCIPAL FACTOR ANALYSIS OF TBMC (SIX FACTOR SOLUTION)

GRADE EIGHT

Items	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5	Factor 6	communality h^2
1			.65*				.50
2			.58*				.35
3			.55*				.35
4			.37*	.33*			.39
5			.51*	.43*			.48
6				.54			.38
7			.35*				.32
8		.43					.38
9							.20
10						.68	.52
11						.69	.50
12				.64			.48
13				.46*			.41
14	.73*						.56
15	.37*						.34
16	.64*						.45
17	.72*						.56
18	.73*						.65
19		.52*					.34
20		.57*					.48
21		.64*					.54
22		.35*		.50*			.43
23		.62*					.49
24		.61*					.45
25							.41
26	.49*						.48
27					.70		.56
28						.56	.35
29					.69		.56
30	.42*	.35*		.37*			.50
% Total variance	10.09	9.07	7.86	6.66	5.78	5.16	
% Common variance	22.61	20.33	17.62	14.92	12.95	11.57	
Sum of communalities							13.39

Total variance accounted for 44.62

*Items included on subtests.

TABLE 25. PRINCIPAL FACTOR ANALYSIS OF FRACTION ACHIEVEMENT TEST (EIGHT FACTOR SOLUTION) GRADE EIGHT

Items	Factor 1	2	3	4	5	6	7	8	Communality h^2
1									.40
2					.41*				.38
3									.35
4	.38*								.36
5									.29
6					.45				.31
7			.43						.38
8						.79			.65
9						.76			.63
10						.72			.54
11					.75*				.62
12			.36*		.48*				.54
13			.40*		.55*				.57
14			.34*		.62*				.65
15			.65*						.64
16			.82*						.75
17			.80*						.78
18					.42*				.25
19			.72*						.67
20			.74*						.60
21					.37		.38		.44
22								.53	.44
23							.41		.43
24						.37			.46
25								.61	.42
26								.65	.46
27								.42	.55
28				.43					.38
29					.41		.35		.58
30	.35								.27
31				.60*			.43		.59
32				.65*					.58
33				.50*					.41
34				.58*					.57
35				.78*					.68
36				.78*					.67
37	.33								.30
38					.40				.23

(Cont'd)

TABLE 25 (Cont'd)

Items	Factor 1	2	3	4	5	6	7	8	Communality h^2
39									.22
40					.34				.35
41						.41			.40
42		.66*							.55
43		.72*							.58
44		.33*							.35
45		.69*							.56
46		.63*							.60
47		.60*							.62
48		.68*							.54
49		.74*							.62
50							.65*		.61
51							.57*		.40
52	.56*								.44
53		.45*			.33*				.50
54	.50*								.38
55		.34*							.21
56	.45*								.32
57	.76		.34*						.50
58	.58*								.47
59	.52*								.45
60	.65*								.57
61	.44*								.42
62	.59*								.50
63	.54*								.41
64	.63*								.49
65	.48*								.46
66	.73*								.62
67	.70*								.56
68	.66*								.53
% Total variance	9.85	7.21	6.68	6.40	6.21	4.23	3.95	3.94	
% Common variance	20.33	14.88	13.78	13.21	12.81	8.72	8.15	8.12	
Sum of communalities									32.96

Total variance accounted for 48.47

*Items included on subtests.

Factor two defines an *addition* subtest with addition items 42 to 49 loading .33 to .74 and items 53 and 55, both subtraction, loading .45 and .34 respectively. The subtraction items are common to both factors one and two but were selected as a logical extension of addition. Items 42 to 55 form the *addition/subtraction* subtest.

Eight of the ten number line items have moderate to high loadings on factor three, and six show high loadings on factor five. The ten items, 11 through 20, form the *number line* subtest.

Factor four defines an *equivalence* subtest composed of items 31 to 36 with loadings of .50 to .78.

A final subtest was selected on the basis of fairly consistent factor loadings for the associated items. Items 1 through 5 showed only two loadings greater than .33 but had similar factor patterns. The items are included in an *initial fraction* subtest.

The subtests resulting from the item selection are:

Subtest 1. Initial fractions — items 1-5.
Subtest 2. Number line — 11-20.
Subtest 3. Equivalence — 31-36.
Subtest 4. Addition/Subtraction — 42-55.
Subtest 5. Multiplication/Division — 56-58.

Relationships exist between subtest 1 and 2 and between subtests 4 and 5.

An examination of the fifteen factor solution (see Appendix C) supports the established subtests. Factors one and seven show high loadings for number line items 11 through 20. Factors four and eight show high loadings for the addition/subtraction items and factors three and four show high loadings for the multiplication/division items.

Factors five and nine include all items for the *equivalence* subtest. Subtest 1 items emerge with high loadings on factors eleven, twelve, thirteen and fifteen. Few relationships exist among the five subtests. Factors one, four and twelve are the only factors with item loadings greater than .33 from two different subtests.

c. Subtests of the Fraction Retention Test

The eight factor solution shown in Table 26 provides the basis for subtest item selection. Factor one with high loadings for items 9 to 15, 17 and 18 defines the *number line* subtest.

Factor two has moderate to high loadings for items 40 through 50, all of which are addition and subtraction items. The items form the *addition/subtraction* subtest.

Factor three shows well defined structure for items 20 to 26, reducing and renaming items, which did not emerge in the Achievement test analysis. The items are excluded from this set of subtests in order that comparisons between canonical correlation results be facilitated.

An *equivalence* subtest includes items 29 through 34 with loadings of .37 to .76 on factor four. Factor six contributes to the subtest with items 29, 30 and 31 having loadings of .68, .42 and .36 respectively.

Items 53 to 56, 58, 60, 61 and 62 have moderate to high loadings on factor five forming a *multiplication/division* subtest. Items 57, 58 and 59 also have high loadings on factor seven. These are included in the subtest.

TABLE 26. PRINCIPAL FACTOR ANALYSIS OF FRACTION RETENTION TEST (EIGHT FACTOR SOLUTION) GRADE EIGHT

Items	Factor								Communality h^2
	1	2	3	4	5	6	7	8	
1						.43*			.25
2	.38*								.59
3									.63
4									.39
5									.31
6						.48			.25
7	.35								.27
8								.36	.30
9	.40*								.53
10	.72*								.62
11	.58*								.45
12	.62*								.53
13	.74*								.59
14	.65*								.57
15	.65*								.56
16				.42*					.46
17	.79*								.65
18	.57*								.48
19						.33	.39		.48
20			.40						.42
21			.77						.68
22			.46			.50			.59
23			.67						.60
24			.61						.42
25			.71	.35					.66
26			.52			.40			.54
27		.33							.49
28						.51			.45
29				.47*		.68*			.74
30				.37*		.42*			.46
31				.59*		.36*			.61
32				.50*					.43
33				.74*					.62
34				.76*					.61
35	.36						.33		.40
36								.52	.37
37								.37	.21
38	.39								.39
39							.49		.43
40		.38*							.28

(Cont'd)

TABLE 26 (Cont'd)

Items	Factor 1	2	3	4	5	6	7	8	Communality h^2
41		.41*							.35
42		.57*							.45
43		.39*							.40
44		.57*			.44*				.61
45		.37*							.31
46		.72*							.65
47	.57*								.47
48	.35*	.41*							.40
49		.66*							.49
50		.50*							.36
51					.35				.18
52									.18
53									.30
54					.58*				.41
55					.66*				.51
56					.58*				.47
57							.65*		.53
58					.50*		.40*		.47
59							.65*		.49
60						.44*			.40
61						.59			.51
62						.44		.34	.45
% Total variance	9.02	6.6	6.0	5.6	5.5	5.0	4.6	3.8	
% Common variance	19.5	14.4	12.9	12.2	12.0	10.8	10.0	8.3	
Sum of communalities									28.7

Total variance accounted for 46.2

*Items included in subtests.

Of the first five items, all initial fraction, three have high negative loadings on factor eight. Once again the factor pattern for the five items is consistent enough to combine them to form an *initial fraction* subtest.

The subtests resulting from the eight factor solution are:

Subtest 1. Initial fractions — items 1-5.

Subtest 2. Number line — 9-18.

Subtest 3. Equivalence — 29-34.

Subtest 4. Addition/Subtraction — 40-50.

Subtest 5. Multiplication/Division — 52-60.

A relationship appears to exist between subtests 2 and 5, 1 and 3, and 1 and 2.

Inspection of the fifteen factor solution (see Appendix C) supports the selection of items for the *number line* subtest (see factor one), *equivalence* subtest (see factor three), *addition/subtraction* (see factors four and six) and the *multiplication/division* subtest (see factors five, eight, eleven, fourteen and fifteen).

Subtest 1, *initial fractions*, has a better factor structure for this solution with high loadings for items 2, 3 and 4, from .63 to .81, on factor seven. Items 1 and 5 have loadings of .20 and .32 respectively.

Relationships that exist between subtests are evidenced on factors one, three, five, nine, ten, eleven and thirteen.

d. Canonical Correlation Analyses Grade Eight

The three TBMC subtests and each set of fraction subtests were submitted to a canonical correlation analysis using the TBMC subtests

as predictors and first the achievement subtests as criterion variables and then the retention subtests.

The hypothesis being tested is:

Hypothesis I. *TBMC subtest scores are independent of fraction subtest scores at the grade eight level*

a) for the Fraction Achievement subtests as criterion variables,

b) for the Fraction Retention subtests as criterion variables.

(i) Fraction Achievement Subtests as criterion variables. A summary of the canonical correlation analysis on the five fraction achievement subtests as criterion variables and three TBMC subtests as predictors is given in Table 27.

The null hypothesis is rejected and the weights associated with the significant canonical correlation are listed at the bottom of Table 27.

The canonical correlation indicates achievement subtests 1, 2 and 5 are most closely related to the three TBMC subtests.

(ii) Fraction Retention Subtests as criterion variables. Results of the canonical correlation analysis using retention subtests are reported in Table 28. The null hypothesis is rejected and the weights associated with the significant correlation coefficient are given.

The significant relationship results from closely related scores on retention subtests 1, 2 and 5 and three TBMC subtests. The *number line* is the most important retention subtest in the relationship with a normalized weight of .81. Subtests 3 and 4, with weights of .08 and .12 make no practical contribution to the relationship.

TABLE 27. CANONICAL CORRELATION ANALYSIS ON ACHIEVEMENT SUBTESTS AND TBMC SUBTEST FOR GRADE EIGHT

Canonical Correlation	Chi Square	df	P
.50	50.1		.000
.21	8.9		.354
.13	2.6		.466

Weights Associated with Significant Correlation

TBMC subtests	*Weight*	*Achievement subtests*	*Weight*
1. Linear measure and subdivision	.55	1. Initial	.55
2. Area measure	.52	2. Number line	.56
3. Area subdivision	.66	3. Equivalence	.33
		4. Addition/Subtraction	.21
		5. Multiplication/Division	.49

TABLE 28. CANONICAL CORRELATION ANALYSIS ON RETENTION SUBTESTS AND TBMC SUBTESTS FOR GRADE EIGHT

Canonical Correlation	Chi Square	df	P
.45	47.1	15	.000
.29	15.2	8	.055
.15	3.0	3	.389

Weights Associated with Significant Correlation

TBMC subtests	*Weight*	*Retention subtests*	*Weight*
1. Linear measure	.62	1. Initial	.44
2. Area measure	.58	2. Number line	.81
3. Area subdivision	.54	3. Equivalence	.08
		4. Addition/Subtraction	.12
		5. Multiplication/Division	.35

Results of the Data Analysis: Question II

Question II. *Do school attended, sex and nonverbal I.Q. affect measurement test (TBMC) scores at the three grade levels?*

The associated null hypothesis is:

Hypothesis II. *There is no significant difference between mean scores on the TBMC*

a) for the three school levels,

b) for the sex categories,

c) for the high/low nonverbal I.Q. categories.

1. *Grade Four ANOVA on TBMC Scores*

Cell means for the grade four TBMC scores are shown in Table 29.

A three way analysis of variance was carried out on the TBMC scores. A summary of results is presented in Table 30.

The school factor F ratio (6.23) is significant at the .01 level and the null hypothesis that there are no differences between mean scores for schools is rejected. The difference between I.Q. level mean scores is significant at the .01 level and the null hypothesis rejected for I.Q. The null hypothesis that no differences exist between sex group means and that there are no significant interactions is not rejected because the F ratios do not exceed the critical values.

Since the school factor shows significant differences between mean scores, the Scheffé test is applied to determine which schools are significantly different. A summary of results is presented in Table 31.

Students from schools one and two perform significantly better than students from school three and students with high nonverbal I.Q. scores perform significantly better than students with low I.Q. scores.

TABLE 29. CELL AND GROUP MEANS ON TBMC GRADE FOUR

		School			
Nonverbal I.Q.	Sex	1	2	3	$\bar{X}$I.Q.
H	M	13.7	17.3	15.3*	15.3
	F	16.2	17.4	13.0	
L	M	13.1	12.9	9.5	12.0
	F	13.4	12.8	9.8	
$\bar{X}$ School		14.3	14.4	11.0	
$\bar{X}$ Sex		13.7(F)	13.3(M)		

*n=3

TABLE 30. ANOVA FOR TBMC GRADE FOUR

Source		SS	df	MS	F	P
school	(A)	134.48	2	67.24	6.23	.003
sex	(B)	.23	1	.23	.02	.884
nonverbal I.Q.	(C)	276.05	1	276.05	25.58	.000
A X B		18.69	2	9.94	.92	.402
A X C		44.45	2	22.22	2.06	.133
B X C		.06	1	.06	.01	.942
A X B X C		18.69	2	9.35	.87	.424
error		1046.77	97	10.79		

TABLE 31. SCHEFFÉ COMPARISONS OF SCHOOL LEVEL MEANS GRADE FOUR

Schools (levels)	F Ratio	P
1-2	.006	> .05
1-3	15.83	< .01
2-3	16.48	< .01

$F_{.01}(2,100) = 2(4.82) = 9.64$
$F_{.05}(2,100) = 2(3.09) = 6.18$

TABLE 32. CELL AND GROUP MEANS ON TBMC GRADE SIX

Nonverbal I.Q.	Sex	School 1	School 2	School 3	$\overline{X}$ I.Q.
H	M	19.7	20.5	19.0	20.3
	F	20.8	21.6	19.8*	
L	M	16.0	17.5	14.9	16.4
	F	18.9	14.8	15.7	
$\overline{X}$ School		19.2	18.6	16.5	
$\overline{X}$ Sex		18.6(F)	18.2(M)		

*n=4

2. *Grade Six ANOVA on TBMC Scores*

Cell means for the grade six TBMC scores are shown in Table 32. Results of the three way analysis of variance carried out on the scores are reported in Table 33.

At the grade six level the null hypothesis is rejected only for the nonverbal I.Q. group means. The F ratio of 28.29 is significant at the .01 level. Mean scores for boys and girls are not significantly different nor are differences between school mean scores. The null hypothesis is not rejected for those factors nor for the effects of interaction.

3. *Grade Eight ANOVA on TBMC Scores*

Cell and group means for grade eight TBMC scores are reported in Table 34. Results of the analysis of variance are reported in Table 35.

The F ratios of 5.84 and 43.16 are highly significant for school and I.Q. respectively. The null hypothesis is rejected in both cases

TABLE 33. ANOVA for TBMC GRADE SIX

Source	SS	df	MS	F	P
school (A)	41.77	2	20.88	1.38	.254
sex (B)	11.93	1	11.93	.79	.376
nonverbal I.Q. (C)	426.95	1	426.94	28.29	.000
A X B	51.03	2	25.52	1.69	.189
A X C	28.33	2	14.16	.94	.394
B X C	2.90	1	2.90	.19	.662
A X B X C	45.48	2	22.74	1.51	.226
error	1886.59	125	15.09		

TABLE 34. CELL AND GROUP MEANS ON TBMC GRADE EIGHT

Nonverbal I.Q.	Sex	School 1	School 2	School 3	$\overline{X}$ I.Q.
H	M	25.5	22.5	22.3	23.7
	F	24.8	22.8	22.2	
L	M	19.2	17.9	18.6	19.0
	F	21.1	19.7	16.4	
$\overline{X}$ School		23.1	20.9	19.2	
$\overline{X}$ Sex		21.5(F)	21.3(M)		

TABLE 35. ANOVA FOR TBMC GRADE EIGHT

Source	SS	df	MS	F	P
school (A)	176.34	2	88.17	5.84	.004
sex (B)	.71	1	.71	.05	.829
nonverbal I.Q. (C)	651.45	1	651.45	43.16	.000
A X B	22.30	2	11.15	.74	.480
A X C	9.67	2	4.83	.32	.726
B X C	3.98	1	3.98	.26	.608
A X B X C	26.21	2	13.11	.87	.422
error	2022.19	134	15.09		

and the Scheffé test applied to the school means to determine which means are significantly different. Results of the Scheffé tests are shown in Table 36.

Results of the statistical tests indicate that school mean for school level 1 is significantly higher than for school levels 2 or 3. No significant difference exists between means for schools 2 and 3.

4. Summary of ANOVA on TBMC Scores at Three Grade Levels

A comparison of the effects of school, sex and I.Q. factors on TBMC scores at the three grade levels shows that nonverbal I.Q. affects TBMC scores at all levels. The school factor has a significant effect at the grades four and eight levels. Boys and girls mean scores are not significantly different at any level. There are no significant interaction effects among the factors.

Results of the Data Analysis: Question III

Question III. *Does performance of students from different schools, having high or low TBMC scores, and high or low nonverbal I.Q. scores, differ on fraction test scores at the three grade levels?*

The associated null hypotheses are:

Hypothesis III (1) *There is no significant difference between mean Fraction Achievement scores*

a) for the different school levels,

b) for high and low nonverbal I.Q. groups,

c) for high and low TBMC groups.

Hypothesis III (2) *There is no significant difference between mean Fraction Retention scores*

a) for the different school levels,

b) for high and low nonverbal I.Q. groups,

c) for high and low measure groups.

TABLE 36. SCHEFFÉ COMPARISONS OF SCHOOL LEVEL MEANS GRADE EIGHT

School levels	F Ratio	P
1-2	8.70	< .05
1-3	20.05	< .01
2-3	4.33	> .05

$F_{.01}(2,125) = 2(4.78) = 9.56$
$F_{.05}(2,125) = 2(3.07) = 6.14$

TABLE 37. CELL AND GROUP MEANS ON FRACTION ACHIEVEMENT GRADE FOUR

		School		
TBMC	nonverbal I.Q.	1	2	$\overline{X}$TBMC
H	H	34.6	28.9	30.8
	L	30.9	27.5	
L	H	29.8	20.0*	23.4
	L	27.7	20.8	
$\overline{X}$ School		21.9	24.3	
$\overline{X}$ I.Q.		30.8(H)	24.2(L)	

*n=3

The hypotheses were tested at the three grade levels. Results are reported below.

1. Grade Four ANOVA on Achievement Scores

The Fraction Achievement test cell and group means are reported in Table 37. A summary of the results of the analysis of variance carried out on the scores is presented in Table 38. The factorial design includes three factors: school (two levels); nonverbal I.Q. (two levels), and TBMC (two levels). F ratios for school and TBMC exceed the critical values and the null hypothesis is rejected for those factors. Differences between mean scores for the two I.Q. groups are not significant nor are any interaction effects. Mean scores for students from school level one and for the high TBMC group are higher than mean scores for students from school level two and from the low TBMC group.

TABLE 38. ANOVA ON FRACTION ACHIEVEMENT SCORES GRADE FOUR

Source		SS	df	MS	F	P
school	(A)	681.69	1	681.69	15.00	.000
nonverbal I.Q.	(B)	43.06	1	43.06	.95	.333
TBMC	(C)	577.38	1	577.38	12.71	.001
A X B		28.41	1	28.41	.63	.431
A X C		61.16	1	61.16	1.35	.249
B X C		15.44	1	15.44	.34	.561
A X B X C		.56	1	.56	.01	.912
error		4589.06	101	45.44		

2. Grade Four ANOVA on Retention Scores

Cell and group means for the Fraction Retention test are reported in Table 39. Results of the three way analysis of variance carried out on the scores are reported in Table 40.

The F ratio exceeds the critical values for school, TBMC, and all interactions. The null hypothesis is rejected for the above cases. Differences between mean scores for students in high and low I.Q. categories are not significant. The null hypothesis is not rejected for I.Q.

With highly related independent variables it is not possible to determine the exact amount of variance accounted for by each factor. In the case of interactions, it can be safely assumed that very little variance is accounted for, and, although the interaction effects are statistically significant, tests are not performed on individual cells. Interactions are shown in Figure 5.

Figure 5a is the plot for nonverbal I.Q. with schools on retention scores. Although high I.Q. students do better than low I.Q. at both school levels, students from school 1 with low I.Q. do nearly as well as students from school 2 with high I.Q. Mean scores for the two groups are 29.7 and 30.1 respectively. The effect of I.Q. seems to be greatest at the school 2 level with the high I.Q. group mean score 30.1 and the low I.Q. group mean 23.0.

The interaction of TBMC levels with school (see Fig. 5b) shows a dramatic difference at the school 2 level. As with the I.Q. groups, students from school 1 having low TBMC scores and students from school 2 having high TBMC scores show similar performance on retention.

TABLE 39. CELL AND GROUP MEANS OF FRACTION RETENTION GRADE FOUR

TBMC	nonverbal I.Q.	1	2	$\overline{X}$TBMC
H	H	35.0	33.0	32.6
	L	31.7	28.6	
L	H	31.7	11.3*	23.2
	L	27.3	20.9	
$\overline{X}$ School		32.6	25.7	
$\overline{X}$ I.Q.		32.3(H)	24.5(L)	

*n=3

TABLE 40. ANOVA ON FRACTION RETENTION SCORES GRADE FOUR

Source		SS	df	MS	F	P
school	(A)	1056.18	1	1056.18	28.56	.000
nonverbal I.Q.	(B)	6.07	1	6.07	.16	.686
TBMC	(C)	1421.58	1	1421.58	38.44	.000
A X B		170.38	1	170.38	4.61	.034
A X C		489.64	1	489.64	13.24	.000
B X C		173.06	1	173.06	4.68	.032
A X B X C		239.12	1	239.12	6.47	.012
error		3735.50	101	361.99		

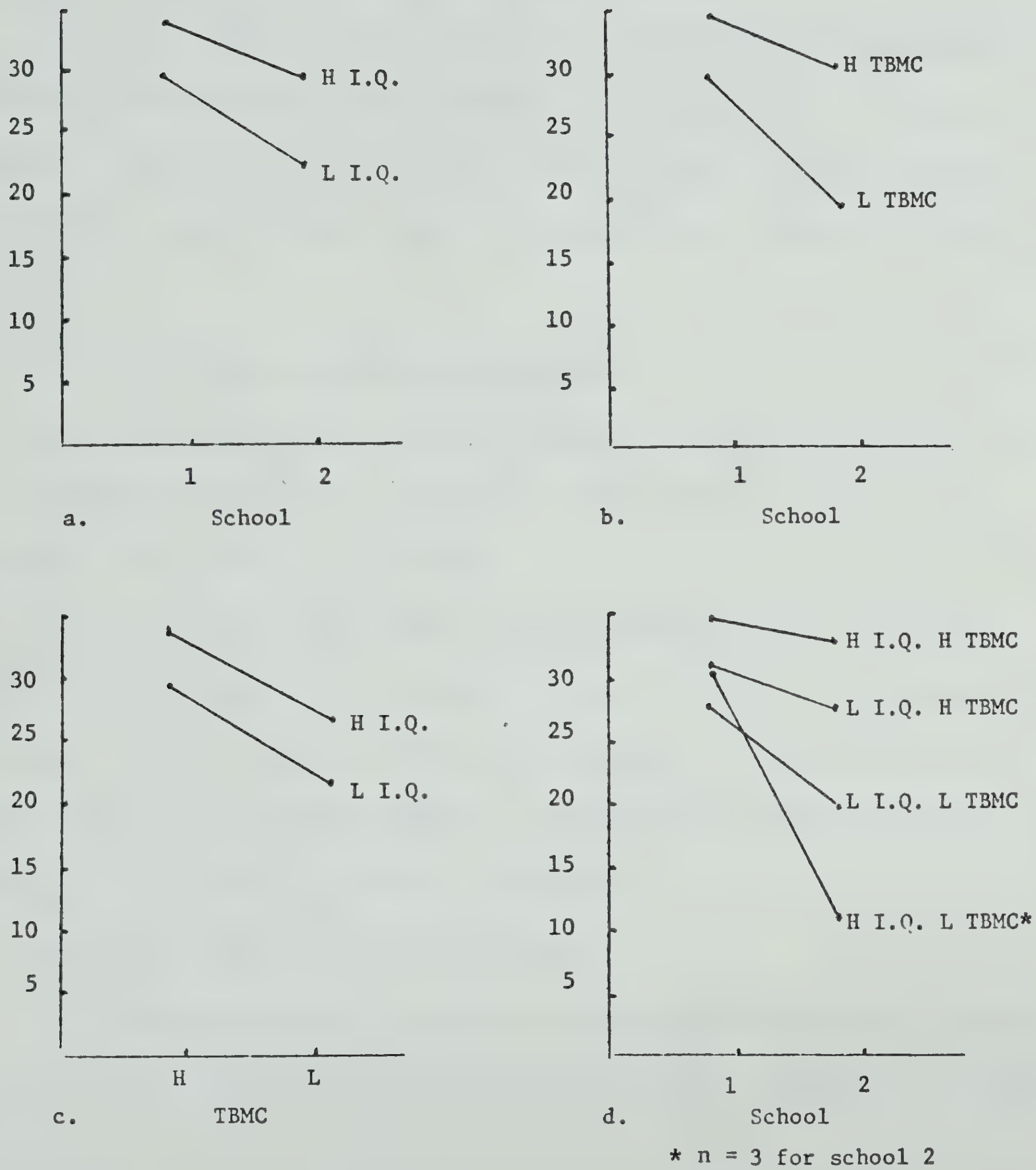


Fig. 5. Interactions among school, I.Q. and TBMC on grade four retention test scores.

Mean scores are 30.1 and 31.3 respectively. The difference between mean scores for students from school 2 having high and low TBMC scores is large. Means are 31.3 and 20.1 respectively.

Figure 5c indicates that students in high TBMC groups outperform students from low TBMC groups regardless of I.Q. level. Figure 5d supports that notion. The plot for H IQ L TBMC on Figure 5d must be viewed with caution since school 2 has only three students in that category.

3. Grade Six ANOVA on Achievement Scores

Cell and group means for the grade six Fraction Achievement test are presented in Table 41. Results of the three way analysis of variance are reported in Table 42.

The F ratios of 56.03 and 14.70 for nonverbal I.Q. and TBMC, respectively, exceed the .01 critical value and the null hypothesis is rejected for both factors. The null hypothesis is not rejected for school and all interaction effects. Students who score low on I.Q. and TBMC also score low on fraction achievement.

4. Grade Six ANOVA on Retention Scores

Cell and group means for grade six retention are given in Table 43. Results of the analysis of variance on the scores are listed in Table 44.

The three main effects are significant as evidenced by the F ratios of 3.92, 35.49 and 15.16 with associated probability levels. The three school levels are submitted to the Scheffé method of comparison to determine which pairs of schools have significantly

TABLE 41. CELL AND GROUP MEANS ON FRACTION ACHIEVEMENT GRADE SIX

		School			
TBMC	Nonverbal I.Q.	1	2	3	$\overline{X}$TBMC
H	H	69.5	70.6	72.8	66.2
	L	56.5	58.8	53.5*	
L	H	66.3	70.0	57.7†	51.4
	L	47.5	44.5	42.6	
$\overline{X}$ School		62.3	60.3	50.6	
$\overline{X}$ I.Q.		69.0(H)	48.7(L)		

*n=2
†n=3

TABLE 42. ANOVA ON FRACTION ACHIEVEMENT SCORES GRADE SIX

Source		SS	df	MS	F	P
school	(A)	203.97	2	101.98	.94	.393
nonverbal I.Q.	(B)	6072.69	1	6072.69	56.03	.000
TBMC	(C)	1593.87	1	1593.87	14.70	.000
A X B		42.34	2	21.17	.20	.823
A X C		146.09	2	73.04	.67	.512
B X C		128.96	1	128.96	1.19	.277
A X B X C		229.29	2	114.15	1.05	.352
error		13548.90	125	108.39		

TABLE 43. CELL AND GROUP MEANS ON FRACTION RETENTION GRADE SIX

TBMC	nonverbal I.Q.	School 1	School 2	School 3	$\overline{X}$TBMC
H	H	74.6	75.9	71.2	70.5
	L	59.8	64.1	58.5*	
L	H	68.8	75.3	55.0†	63.3
	L	51.0	50.9	40.3	
$\overline{X}$ School		66.3	65.9	48.8	
$\overline{X}$ I.Q.		72.8(H)	51.2(L)		

*n=2
†n=3

TABLE 44. ANOVA ON FRACTION RETENTION SCORES GRADE SIX

Source		SS	df	MS	F	P
school	(A)	1154.61	2	577.30	3.92	.022
nonverbal I.Q.	(B)	5226.71	1	5226.71	35.49	.000
TBMC	(C)	2232.31	1	2232.31	15.16	.000
A X B		54.78	2	27.39	.69	.830
A X C		332.45	2	166.23	1.13	.327
B X C		177.16	1	177.16	1.20	.275
A X B X C		18404.60	125	147.24		

different mean scores. Results of the Scheffé tests are reported in Table 45.

Once again students with high nonverbal I.Q. and high TBMC scores show higher mean scores on retention tests. School levels 1 and 2 have significantly higher means than the mean retention score for school level 3.

None of the interaction effects are significant at the .05 level.

TABLE 45. SCHEFFÉ COMPARISONS OF SCHOOL LEVEL MEANS FRACTION RETENTION GRADE SIX

School levels	F ratio	P
1-2	.02	$> .05$
1-3	42.8	$< .01$
2-3	34.9	$< .01$

$F_{.01}(2,125) = 2(4.78) = 9.56$
$F_{.05}(2,125) = 2(3.07) = 6.14$

5. *Grade Eight ANOVA on Achievement Scores*

Cell and group means for grade eight achievement are reported in Table 46. The analysis of variance results are reported in Table 47.

The null hypothesis that there are no significant differences between means for school levels is accepted. F ratios of 15.3 and 6.72 for I.Q. and TBMC exceed the critical value and the null hypothesis

TABLE 46. CELL AND GROUP MEANS FOR FRACTION ACHIEVEMENT GRADE EIGHT

TBMC	Nonverbal I.Q.	School 1	School 2	School 3	$\overline{X}$TBMC
H	H	62.7	59.7	59.3	59.9
	L	57.3	54.2	52.0	
L	H	52.3*	58.6	60.4	50.2
	L	46.9	45.5	50.8	
$\overline{X}$ School		56.6	53.9	53.9	
$\overline{X}$ I.Q.		60.5(H)	49.3(L)		

*n=3

TABLE 47. ANOVA ON FRACTION ACHIEVEMENT SCORES GRADE EIGHT

Source		SS	df	MS	F	P
school	(A)	20.88	2	10.44	.12	.887
nonverbal I.Q.	(B)	1329.82	1	1329.82	15.27	.000
TBMC	(C)	584.73	1	584.73	6.72	.010
A X B		64.27	2	32.13	.37	.692
A X C		356.08	2	178.04	2.05	.133
B X C		60.21	1	60.21	.69	.407
A X B X C		65.32	2	32.66	.38	.688
error		1664.40	134	87.05		

is rejected. Students in the high nonverbal I.Q. and high TBMC categories have significantly higher mean achievement scores than students in low I.Q. and low TBMC categories.

6. Grade Eight ANOVA on Retention Scores

Cell and group means for grade eight retention are reported in Table 48. Results of the analysis of variance on the retention scores are reported in Table 49.

The F ratios for nonverbal I.Q. and TBMC are significant at the .01 and .05 levels respectively. Mean retention scores for students having high I.Q. or high TBMC scores are higher than students with low I.Q. or TBMC scores. Differences between retention means for students in the three school categories are not significantly different. The null hypothesis is rejected for I.Q. and TBMC but not rejected for school.

TABLE 48. CELL AND GROUP MEANS FOR FRACTION RETENTION GRADE EIGHT

		School			
TBMC	nonverbal I.Q.	1	2	3	$\overline{X}$TBMC
H	H	58.2	56.2	58.7	56.7
	L	56.8	49.8	54.0	
L	H	55.0*	57.2	55.6	50.6
	L	47.8	47.5	51.2	
$\overline{X}$ School		54.6	53.7	53.6	
$\overline{X}$ I.Q.		57.3(H)	49.9(L)		

*n=3

TABLE 49. ANOVA ON FRACTION RETENTION SCORES GRADE EIGHT

Source		SS	df	MS	F	P
school	(A)	91.11	2	45.55	.96	.387
nonverbal I.Q.	(B)	705.80	1	705.80	14.81	.000
TBMC	(C)	233.45	1	233.45	4.89	.029
A X B		73.31	2	36.65	.77	.465
A X C		128.08	2	61.54	1.29	.278
B X C		49.13	1	49.13	1.03	.312
A X B X C		30.59	2	15.29	.32	.726
error		6384.44	134	47.65		

7. *Summary of ANOVA on Achievement and Retention Scores at the Three Grade Levels*

Mean achievement and mean retention scores between high and low TBMC groups are significantly different at all grade levels. Mean scores between schools are significantly different at the grade four level for achievement and retention and at the grade six level for retention. Mean scores between high and low nonverbal I.Q. groups are significantly different at the grade six and eight levels for achievement and retention.

Due to the high relationships among the independent variables it is not possible to assess the amount of variance accounted for by the three factors in each analysis. It appears that TBMC contributes most consistently to the differences in scores for both achievement and retention tests.

Results of the Data Analysis: Question IV

Question IV. *Is the TBMC a potentially useful predictor of rational number achievement at the three grade levels?*

The question is answered partly through determination of statistical significance of correlation coefficients and partly through a determination of how much variance is accounted for by the relationship. The first is achieved through an examination of correlation coefficients for the fraction tests with TBMC and I.Q. The hypothesis being tested is:

Hypothesis IV. *The correlations between TBMC scores and tests of fractional concepts are not significantly different from 0.*

If the null hypothesis is rejected for any of the considered coefficients, then the predictor is compared with nonverbal I.Q. as a traditional predictor of achievement and variance accounted for by each relationship reported.

1. Grade Four Correlations

The correlation coefficients among the fraction test scores, TBMC scores, and nonverbal I.Q. scores at the grade four level are shown in Table 50.

Both coefficients between TBMC and fraction tests are significantly different from 0 at the .01 level. Comparing these with the correlation coefficients between nonverbal I.Q. and the fraction tests shows the I.Q. with achievement coefficient, .49, is slightly higher than TBMC with achievement, .47. The I.Q. with retention coefficient, .54, is slightly lower than TBMC with retention, .56. Variance accounted

TABLE 50. CORRELATION MATRIX FOR GRADE FOUR*

	1	2	3	4
1. Fraction achievement	1.0			
2. Fraction retention	.85	1.0		
3. TBMC	.47	.56	1.0	
4. Nonverbal I.Q.	.49	.54	.53	1.0

*All correlations significant at the .01 level.

for by the relationships is approximately 22% and 31% for TBMC with achievement and retention, respectively, and 24% and 29% of nonverbal I.Q. with achievement and retention.

The comparisons suggest that TBMC scores are at least as good as I.Q. in predicting fraction achievement at the grade four level.

2. *Grade Six Correlations*

The correlation coefficients among the two fraction tests, TBMC, and nonverbal I.Q. are reported in Table 51. All coefficients are significantly different from 0 at the .01 level.

The correlation coefficients between TBMC and achievement and retention are .55 and .53 respectively. The relationships account for 30% and 28% of the variance for the respective variables. Correlation coefficients between nonverbal I.Q. and the fraction tests are .72 and .73 and account for 52% and 53% of the variance of achievement and retention respectively.

At the grade six level, nonverbal I.Q. is the better predictor of achievement and retention scores.

TABLE 51. CORRELATION MATRIX FOR GRADE SIX*

	1	2	3	4
1. Fraction achievement	1.0			
2. Fraction retention	.91	1.0		
3. TBMC	.55	.53	1.0	
4. Nonverbal I.Q.	.72	.73	.50	1.0

*All correlations significant at the .01 level.

3. *Grade Eight Correlations*

The correlations among the two fraction tests, TBMC, and nonverbal I.Q. are reported in Table 52.

TABLE 52. CORRELATION MATRIX FOR GRADE EIGHT*

	1	2	3	4
1. Achievement	1.0			
2. Retention	.79	1.0		
3. TBMC	.48	.38	1.0	
4. Nonverbal I.Q.	.62	.59	.55	1.0

*All correlations significant at the .01 level.

All correlation coefficients are different from 0 at the .01 level of significance. Coefficients between I.Q., and achievement and retention, are higher than coefficients between TBMC and the fraction tests.

Variance accounted for by the nonverbal I.Q. relationships with criterion variables are 38% and 35% for achievement and retention respectively. TBMC scores account for 23% and 14% respectively. Of the two, nonverbal I.Q. is the better predictor of fraction test scores.

4. *Summary of Correlation Results at the Three Grade Levels*

Correlation coefficients derived from TBMC scores and the two criterion variables are significantly greater than 0 at all grade levels. Comparisons with correlation coefficients between nonverbal I.Q. and the two criterion variables show no practical difference between predictive potential for TBMC and nonverbal I.Q. at the grade four level. At the grade six and eight levels nonverbal I.Q. appears to be the superior predictor.

Correlation coefficients computed on total test scores for the TBMC and the fraction tests vary from the canonical correlations computed on linear combinations of subtests for the variables. A discussion of the differences and their implications is presented in the last chapter.

Chapter VI

SUMMARY, CONCLUSIONS AND IMPLICATIONS FOR FURTHER STUDY

Introduction

The present study has two purposes. One is to determine the relationship between linear and area basal measurement concepts and fraction number concepts. The other is to determine the level at which students in grades four, six and eight are functioning on linear and area measurement tasks. The purposes are achieved through application of four questions and the associated statistical hypotheses.

This chapter provides a summary of research results. First, the results are summarized under the main question headings and conclusions are drawn. Interpretations of results are included. Next, a section is devoted to suggestions for further study including experiments based on findings of this research. Finally, the results of this study are evaluated in terms of the purposes of the study and related to findings from other research studies.

Summary of Results and Conclusions

Question I. *What is the nature of the relationship between measurement concepts and rational number concepts at the three grade levels?*

Results of the canonical correlation analyses using Fraction Achievement and Fraction Retention subtest scores at three grade levels were reported in Chapter V. The results are summarized as follows:

1. The *initial fraction* subtest contributes most consistently to the relationship between TBMC and fraction subtests at all levels.
2. *Equivalence* subtests make low to moderate contributions to the significant relationship at the grades four and six levels on both analyses and low or no contribution at the grade eight level.
3. *Number line* subtests make no contribution to the significant relationships on the grade six achievement and retention analyses and grade four achievement analysis, and a very weak contribution on the grade four retention analysis.
4. *Number line* subtests make the major criterion contribution to the significant relationship on the grade eight analyses.
5. *Multiplication/Division* subtests make essentially no contribution to the significant relationship on the grade six analyses and low contributions on the grade eight analyses.
6. *Addition/Subtraction* subtests make moderate to low contributions to significant relationships on grade six analyses and low contributions on grade eight analyses.
7. Subtests of the TBMC, involving *area* or *linear subdivision*, make high contributions to significant relationships among subtests at all levels.
8. *Area measure* subtests make high contributions to significant relationships at all grade levels.
9. *Linear measure* subtests are most important at the grade 8 level, make a moderate contribution at grade 6 and no contribution at grade 4.

10. The amount of variance accounted for by the significant canonical correlations was greater for the analysis involving retention subtests than for achievement subtests at the grade four and six levels. Variance accounted for ranges from 23% and 28% on the achievement subtest analyses to 35% and 34% on the retention subtest analyses for grades four and six respectively. At the grade eight level variance accounted for ranges from 25% on achievement analysis to 20% on retention.

A number of conclusions can be drawn from the above results. It would appear that the TBMC subtests are strongly related to the *initial fraction* subtest at all grade levels. Initial fraction as used here refers only to area (region) concepts. In particular, the *subdivision* subtests and *area measure* subtests make high contributions to relationships at all levels. One conclusion to be drawn is that the ability to operate in terms of area unit iteration and linear and area subdivision is necessary for success with initial fraction concepts. If one assumes that initial fraction concepts are fundamental to the learning of later fraction concepts then the implication is that understanding of area and linear concepts is necessary for fraction learning to take place. The basic behavioral statement is that students who can operate with basal measurement tasks can also operate with initial fraction tasks.

The contribution made to the relationship by the *number line* subtest at the three grade levels is a most important consideration here. Performance on *linear* subtests does not appear to be related to

performance on number line items until the grade eight level. This result suggests that students who are able to operate with linear measure concepts do not associate the operations with number line operations at the grade four and six levels. At the grade four level, the *linear measure* subtest makes a very small contribution to the significant relationship among variables. This would suggest that grade four students, although theoretically able to operate on linear measure tasks, are not generalizing that ability to the number line. At the grade six level linear measure is operational and a contribution is made to the relationship but not specifically to the *number line* subtest. This is a rather surprising result considering the notion of linear measure items as a more general case of number line. Students at the grade four and six levels apparently do not make the association. Only at the grade eight level does a strong relationship make its appearance. At this level the relationship is the single most important one under study. The results involving number line support the grade four and six teachers' contention that students are not able to understand number line concepts and applications.

The low but consistent contribution to the significant relationship by *addition* and *subtraction* subtests suggests a weak dependence upon measure concepts. The *multiplication/division* subtests make no contribution to the significant relationship at the grade six level. The moderate contribution by *multiplication* and *division* at the grade eight level makes no theoretical sense unless students at this level are able to view area measure in the most advanced sense as a multiplicative two dimensional entity. Such an interpretation is not

necessary for development of area on the unit iteration basis peculiar to fraction models.

The strength of the relationship between measure tasks and fraction achievement and retention concepts increases with the intervention of time at the grades four and six levels. The significant canonical correlations for both grades, using achievement subtests as a criterion, were smaller than significant canonical correlations using retention subtests as a criterion. The TBMC subtest scores accounted for a larger portion of the variance in retention subtest scores than achievement subtest scores. This result could be due to the effects of memory masking the relationship between TBMC subtests and achievement subtests. Retention subtests, under such an assumption, would reflect a truer relationship with TBMC subtests. The canonical correlation dropped slightly from achievement to retention analysis at the grade eight level. Since grade eight fraction units lasted only a few days and no new material was presented, it may be that both fraction tests are acting as retention tests.

In Chapter II it was shown that a theoretical relationship exists between measure concepts and fraction number concepts. Results of the canonical correlation analysis indicate that a relationship exists between measure and fraction number behaviors as assessed by a paper and pencil test. Specifically, initial fraction concepts and the operations of addition and subtraction are related to area measure and linear/area subdivision concepts at all grade levels. Linear measure and number line concepts are strongly related at the grade eight level only.

Kieren (1977) suggested that the measure construct is fundamental to a beginning understanding of rational number. The relationship of the *area measure* and *linear/area subdivision* subtests with *initial fractions* at early levels of fraction instruction supports that notion. It is apparent from this study that measure ability is associated with early success on initial fraction tasks and the relationship continues into junior high school.

Question II. *Do school attended, sex, and nonverbal I.Q. have an effect on TBMC scores at the three grade levels?*

The effects of school, sex, and nonverbal I.Q. on TBMC scores were studied through analysis of variance techniques at the three grade levels. Results of the analyses are reported below.

1. There are no significant differences between TBMC mean scores of boys and girls at the three levels.
2. At the three grade levels, students in the high I.Q. group score significantly higher than students in the low I.Q. group.
3. At the grade four level students from schools one and two scored significantly higher than students from school three.
4. There are no significant differences between school means at the grade six level.
5. At the grade eight level students from school one scored significantly higher than students from schools two and three.

The results suggest that I.Q. affects TBMC scores at all grade levels and sex does not. Mean scores for girls and boys respectively are 13.9 and 13.3 at the grade four level, 18.6 and 18.2 at the grade six level and 21.5 and 21.3 at the grade eight level. Girls score slightly higher at all levels but differences are neither statistically nor practically meaningful. TBMC scores are dependent upon grade level with development of boys and girls scores showing a remarkable similarity. Figure 6 shows a plot of TBMC scores against grade level for boys and girls. A slight levelling off appears to take place between grades six and eight for both boys and girls, but grade differences remain striking.

The significance of the school and I.Q. main effects could be the result of a high relationship between the independent variables. In other words, differences between mean scores for schools could be a function of large numbers of high I.Q. students at one school level and low I.Q. students at another level. Examination of the I.Q. means for the different school levels shows the strongest relationship existing at the grade six level with a difference of thirteen I.Q. points between school level one and school level three. Differences between TBMC mean scores are not significant at this level. At the grade four level school level two has the highest TBMC score, 14.4, and a mean I.Q. score of 103. School level one has mean TBMC 14.3 and mean I.Q. 110. It is concluded that although school and I.Q. are not independent measures, the effect of school on TBMC scores is not simply a redundancy of I.Q. effect.

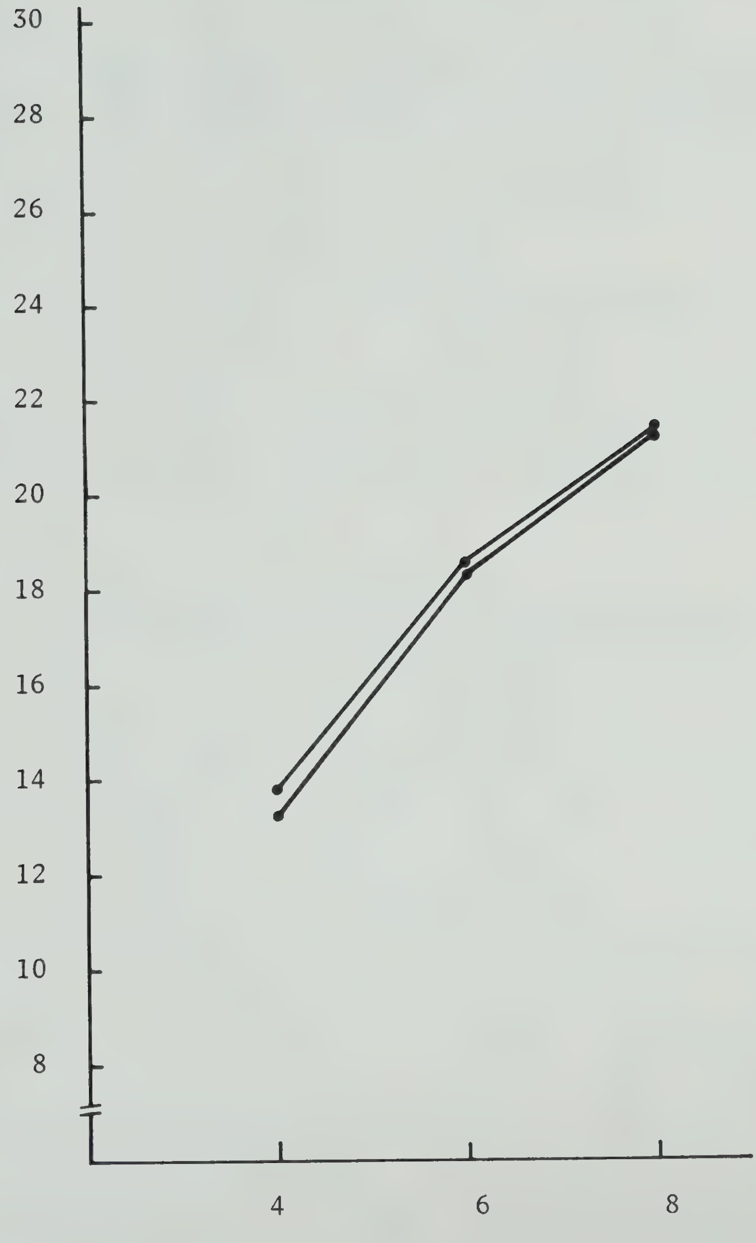


Fig. 6. TBMC scores against grade level for girls and boys.

Question IIa. *What is the nature of differences between measure behaviors, as assessed by the TBMC, of students at the three grade levels?*

Differences between students at the three grade levels are placed in two categories: those associated with the derived TBMC subtests and those associated with items not included in subtests. The first discussion involves items included in subtests.

Items 1 through 5, comparing polygonal paths, increase in difficulty (complexity) from comparison of paths composed of equal sized units to comparisons of paths composed of varying unit sizes. The difficulty index for the five items is shown in Table 53 for the three grade levels.

Each index indicates the proportion of students getting the item correct at each level. The difficulty index increases for all items from grade four to grade eight. Items 4 and 5 are well below an acceptable level for grade four with only 28% and 35% of students responding correctly. Both items involve varying unit segments in both polygonal paths of a pair. Item 1 is the only item grade four students find easy. It involves equal units on both paths. The only requirement is that students be able to count and compare numbers. Grade six and eight students have difficulty with items 4 and 5. At the grade six level approximately half of the students are unable to combine size of unit and number of units successfully. Grade six and eight students are successful on items 2 and 3. Both items involve comparing pairs in which one path is composed of equal units. The other path is composed of equal units which are longer than those in the first path.

TABLE 53. DIFFICULTY INDEX FOR ITEMS 1 THROUGH 5 (TBMC) GRADES 4, 6 AND 8

	Grade level		
Item	4	6	8
1	.71	.76	.86
2	.62	.86	.84
3	.57	.67	.74
4	.28	.47	.56
5	.35	.53	.62

If linear unit iteration is operational at age eight - nine (Coxford 1968), then one would expect most students in grade four to complete the five items satisfactorily. This is clearly not the case. In fact, grade six students are successful on only three of the five items using .65 as a minimum index of acceptability. Most students experience difficulty on the most complex comparisons.

Items 6 and 7 require students to subdivide a line segment into five and six equal parts. Table 54 lists the difficulty indices for the three grade levels.

TABLE 54. DIFFICULTY INDEX FOR ITEMS 6 AND 7 GRADES 4, 6 AND 8

	Grade level		
Item	4	6	8
6	.37	.68	.80
7	.30	.69	.90

Students aged eight years are expected to perform subdivisions successfully on a trial and error basis. The indices of .37 and .30 indicate that this is not the case at the grade four level, where most students are unable to complete the tasks satisfactorily. At this level many students equate number of equal parts with number of partitions, thus ending up with one more than the desired number of subsegments. At the grade six level, student performance is generally satisfactory on the items.

Items 12 and 13 are two simple unit iteration items requiring students to construct segments twice and five times as long as a given segment. Students between ages eight and nine years are expected to master these tasks. Table 55 lists the difficulty indices for the two items at the three grade levels.

TABLE 55. DIFFICULTY INDEX FOR ITEMS 12 AND 13 GRADES 4, 6 AND 8

	Grade level		
Items	4	6	8
12	.65	.85	.93
13	.48	.69	.81

Students at the grade four level experience difficulty with item 13. By grade six an acceptable number of students have mastered the tasks.

Examination of the factor loadings for the linear items discussed so far show an interesting trend for grades four through eight. At

the grade four level items 6, 7, 12 and 13 show a strong relationship in that they have similar factor patterns. The items, which involve subdivision and simple unit iteration, are distinct from items 1 through 5 in the factor solutions. At the grade six level, items 6, 7, 12 and 13 appear to be related to area subdivision items. Not until grade eight do factor structures include combinations of items of linear subdivision, simple unit iteration, and the more complex operational linear measure items, 1 through 5. This trend suggests a hierarchy of development which, in combination with difficulty indices, indicates students at the grade four level have not mastered the simplest linear subdivision and unit iteration tasks as measured by this paper and pencil test. It also appears that the operational concept of linear measure, including the subconcepts simple unit iteration and subdivision, does not crystallize until the grade eight level.

Items 14 through 18 require comparisons of regions composed of unit squares. Area unit iteration theoretically becomes operational between eight and ten years of age. Difficulty indices for items 14 through 18 are presented in Table 56.

More than half of the students in grade four completed each of the tasks successfully. Grade six and eight students found the tasks easy. None of the comparisons involved unequal sized units in a given region. A higher level of complexity exists which could present a greater challenge to students at all levels.

Items 19 through 24 involve subdivision of plane figures into five and six equal parts. Plane figures include squares, circles and rectangles. Difficulty indices are listed in Table 57 for grades

four, six and eight.

TABLE 56. DIFFICULTY INDEX FOR ITEMS 14 - 18, GRADES 4, 6 AND 8

	Grade level		
Item	4	6	8
14	.71	.83	.93
15	.61	.88	.90
16	.69	.80	.93
17	.56	.77	.89
18	.58	.80	.91

TABLE 57. DIFFICULTY INDEX FOR ITEMS 19 - 24, GRADES 4, 6 AND 8

	Grade level		
Item	4	6	8
19 (S)*	.16	.38	.39
20 (C)	.08	.21	.51
21 (R)	.24	.69	.69
22 (S)	.14	.43	.52
23 (C)	.07	.23	.52
24 (R)	.17	.45	.62

*(S) (C) (R) denote square, circle, rectangle items respectively.

All tasks are very difficult for the grade four and six students. At all levels subdivision of the circle is most difficult. This is curious given that circles are used so extensively with fractions.

Squares are next in difficulty. A common error at the grades four and six levels included dividing circles and squares into four equal parts and then subdividing one of the quarter sections to get five parts. A number of students recognized the problem of inequality of subdivisions but suggested that it is impossible to get five equal parts. Another common error was subdivision by placing five or six partitions, the resulting number of parts being one greater than requested. This problem arose with squares and rectangles.

Another common error was placing chords on the circles parallel to a diameter and equidistant from one another. This was occasionally in combination with too many chords to produce the correct number of subdivisions.

At the grade eight level many students interpreted the tasks as requiring a greater degree of sophistication than was intended. Many students did not finish the tasks but pencil work indicated a great number of algebraic techniques had been tried.

Marking of the area subdivision items was very generous in terms of "sloppy" work getting credit. Rationalization for easy marking is that students may be able to recognize correct subdivisions but poor eye - hand co-ordination prevents success in performing the tasks. This problem was confounded at the grade six level where a small number of students took the "jigsaw puzzle" approach to subdivision. This approach could mean that grade six students realize that subdivisions of area need not be congruent to be equal. Unfortunately, the results do not indicate the kind of reasoning that led to solutions. Mastery of subdivision tasks theoretically takes place between eight and ten

years of age. Using performance on pencil and paper tasks as a criterion, *even very low mastery is not apparent at that age.*

The following items were not selected for inclusion on subtests. Two sets of items based on two area objectives are discussed. Items 8 through 11 were selected to test student ability to compare the area of two regions with no units provided. Results of the factor analysis at the grade four level indicate the four items are distributed over factors with very little pattern similarity. Results of the grade six factor analyses show the items loading on two factors, items 8 and 9 on one factor and items 10 and 11 on another. At the grade eight level the items again load on two factors with very high loadings. The difficulty indices presented in Table 58 indicate that area comparison items are generally not difficult at the three grade levels. Item 9 is of particular interest because the difficulty index is low for all grade levels. The item involves comparing a fat with a skinny region. The other items involved either two fat or two skinny regions. Students who focus on length for item 9 might be expected to select the skinny and long region as having greatest area. That answer is incorrect. The incremental increase in difficulty index for item 9 across grade levels is consistent with a developing ability to cope with such perceptual miscues. That only 51% of the grade eight students answered the item correctly is surprising. Items 8 through 11 were constructed to reflect a high level of measure ability, theoretically acquired around ten years of age.

Perhaps the most interesting responses were those made to items 25 through 30. The items require comparing shaded subdivisions of

pairs of regions. The item difficulty index for each item is given in Table 59.

TABLE 58. DIFFICULTY INDEX FOR ITEMS 8 - 12. GRADES 4, 6 AND 8

	Grade level		
Item	4	6	8
8	.73	.80	.78
9	.30	.41	.51
10	.80	.88	.88
11	.62	.59	.70

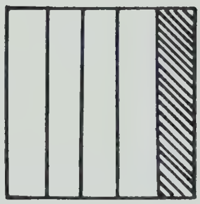
TABLE 59. DIFFICULTY INDEX FOR ITEMS 25 - 30. GRADES 4, 6 AND 8

	Grade level		
Item	4	6	8
25	.65	.74	.78
26	.25	.47	.71
27	.55	.42	.56
28	.37	.44	.60
29	.52	.47	.53
30	.39	.54	.69

Items 27 and 29 provide an interesting contrast to the others. All items except 27 and 29 show decreasing difficulty with grade level. Grade four students found the two items to be relatively easy. Item

27, requiring comparison of shaded parts, is illustrated here. Item 29 is similar.

A

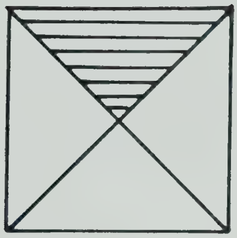


B

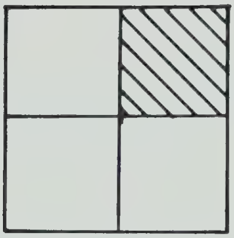
Not only did grade four students find the items only slightly more difficult than grade eight students, the low scoring students in grade four gave proportionally more correct responses than did other grade four students. About two thirds of the poor scorers (of the total TBMC) responded correctly compared with approximately two fifths of the high scorers. At the grade six level students who score low on the total test also score low on the two items. At all grade levels the two items loaded on a single factor. The most compelling conclusion is that many grade four students are focussing on the perceptual attribute of length. Grade six students appreciate the need for logical intervention but most students are unable to reason through to a correct answer. It would seem that grade six students are no longer willing to rely upon what may be perceptually obvious to grade four students.

Items 26 and 30 are a challenge to all but the grade eight students. The required comparison of shaded subdivisions is illustrated in item 26 shown below. Item 30 is similar. The two items load on a single factor for grade six and eight students. At the grade eight

A



B



level the factor structure includes other area measure items, in particular items which test area unit iteration. The trend seems to be from a fairly random distribution on factors of items 25 to 30 for grade four students to a narrowing on factors at the grade six level. At the grade eight level results of the factor solutions suggest that the various area items are combining on factors. This can be interpreted as a crystallization of area measure at the grade eight level.

Question III. *Does performance of students from different schools having high or low I.Q. scores, and high or low TBMC scores, differ on fractional number test scores at the three grade levels?*

Conclusions drawn from results of the analysis of variance on fraction test scores at the three grade levels are listed below.

1. Students in the high mean TBMC group have significantly higher mean fraction achievement and retention scores than students in the low mean TBMC group at all grade levels.
2. Students in high mean I.Q. groups score significantly higher than students in low mean I.Q. groups at the grade six and eight levels.
3. Mean scores between high and low I.Q. groups at the grade four level are not significantly different.

4. Mean scores between school levels are significantly different at the grade four level.
5. Mean scores between school levels are significantly different for grade six retention tests only.

The interdependency among factors causes some difficulty in interpreting main effects. It is possible that TBMC categories are simply reflecting high and low I.Q. scores in grades six and eight. At the grade eight level this is possible since the three school levels have mean I.Q. scores 110, 112 and 116, while mean TBMC scores are 19, 20 and 23 respectively. At the grade six level, however, the four participating schools have mean I.Q. scores 103, 111, 113 and 116 and mean TBMC scores 16, 18, 19 and 17 respectively. Although clearly related, these are not simply redundant measures.

Establishing the importance of each factor depends upon determining not only the statistical significance of the factor contributions but also the amount of variance of criterion scores accounted for by each factor. In view of the relationships among independent variables at all levels, as evidenced by unequal cell frequencies, it is not possible to assess the proportion of variance attributable to each factor. Contributions of factors are assumed to be substantial in view of the large mean squares associated with the significant F ratios. It is assumed that the greatest contributions to variance are from I.Q. followed by TBMC at the grades six and eight levels. At the grade four level school appears to account for most of the variance of achievement scores and TBMC accounts for most of the variance of retention scores. It is believed that substantial contributions to

variance are made by all significant main effects, but it cannot be determined what portion is attributable to each factor.

General conclusions based on analysis results are that TBMC, I.Q. and school are significant main effects on fraction test scores.

Question IV. *Is the TBMC a potentially useful predictor of rational number achievement at the three grade levels?*

The strategy for answering this question involved first computing the correlation coefficient between TBMC scores and fraction test scores at the three grade levels. Significant coefficients were then compared with nonverbal I.Q. correlation coefficients with the same criterion test scores. Conclusions drawn from the correlation coefficients among TBMC and nonverbal I.Q. with the two fraction tests are listed below.

1. Total TBMC scores correlate significantly with fraction test scores at the three grade levels. Variance accounted for ranges from 14% of grade eight retention to 34% of grade four retention.
2. TBMC scores and nonverbal I.Q. scores are equally good predictors of fraction achievement at the grade four level.
3. Nonverbal I.Q. scores are better predictors of fraction test scores than TBMC scores at the grade six level. I.Q. accounts for approximately 20% more variance than does TBMC.
4. Nonverbal I.Q. is a better predictor than TBMC scores of fraction scores at the grade eight level. I.Q.

accounts for approximately 15% to 20% more of the variance than does TBMC.

The correlation results were derived from total TBMC scores with total fraction scores as a gross measure of the predictive potential of the TBMC. The canonical correlation analysis provides a set of correlation coefficients based on subtests selected for interpretability and independence. The canonical correlations provide a measure of predictive potential of the TBMC as a set of separate variables. The significant canonical correlation accounts for approximately the same amount of variance in criterion scores as did the coefficient for total TBMC scores with total criterion scores at the grade four and six levels. At the grade eight level, the significant canonical correlations account for 2% to 6% more variance in criterion scores than does the total TBMC with criterion.

An advantage to using canonical correlation is that the source of variance can be studied. Only one significant canonical correlation resulted for each analysis at each grade level. One source of variance, distinguished by a unique system of weights, permits identification of the most important subtests to the predictive relationship. Specifically, it can be seen (see Tables 18, 23, 28) that the *number line* subtest scores are not predicted by TBMC subtest scores at the grade four and six levels. At the grade eight level, a large portion of the relationship involved the *number line* subtest and the *linear measure* subtest. Inspection of correlation coefficients between total test scores does not afford analysis of such explicit sources of variance.

The predictive potential of the TBMC is viewed not just as a total score potential, but as a contribution of TBMC subtests and criterion subtests to an interrelationship that changes from grade level to grade level. In this sense, the TBMC subtests are viewed as a specialized set of predictors.

Relationship With Other Research Findings

Several studies involving rational number instruction were discussed in Chapter II. Results obtained in this study are related to some of those findings. Novillis (1976) found that students at the grade seven level (prior to rational number instruction) were unable to locate fractions on a number line with equal ease when the number line was of length one and of length two. She also found that students experienced greater difficulty when the unit was divided into more equivalent segments than the number shown in the denominator of the fraction being located. Responses from grade six students (after rational number instruction) from the present study are consistent with Novillis' results. Response by grade eight students indicates mastery of number line tasks. It is significant that the canonical correlation analysis at the grade six level shows no relationship between number line scores and linear measure scores whereas at the grade eight level, the relationship is most important. The evidence supports Novillis' suggestion that students must be made aware that the unit is dependent upon location of the numeral 1 and not the length of the number line. Grade six students do not experience much greater difficulty than grade eight students with linear measure tasks. With appropriate

instruction, it is expected that transfer could take place at the grade six level.

Owens (1977) found that students in grades three and four who scored high on area measure tasks had significantly higher mean scores on fraction concepts than students who scored low on area measure tasks. Results of the present study support that finding at the grade four level. Owens found no difference between grade three and grade four performance on fraction concepts and very little difference on area measure tasks. Data from the present study suggest that grade four students achieve very low with area measure tasks and it is possible that measure development does not have a significant effect with grade level until after grade four.

Bailey (1974) studied success with polygonal path comparisons in children through grade three. Only three of the ninety students tested were able to compare paths with differing subsegments. Each path contained equivalent subsegments but one path had longer segments than the other path. The three successful students were at the grade three level. Results from the present study indicate that 62% of grade four students satisfactorily completed a similar task. Tasks of a more complex nature showed nonmastery at the grade four level. Bailey's contention that the ability to use the dimensions length of unit and number of units simultaneously is not apparent until after age nine is supported. The further suggestion that number line interpretations of rational number are not meaningful to these children is also supported by results from the present study.

Payne (1976) found several persistent difficulties with pilot studies involving the Initial Fraction Sequence (IFS). Number line tasks, area measure concepts, and comparison between fraction symbols proved difficult for students in grades three through five. He concluded that fraction concepts take longer to teach than is generally thought. Results from the TBMC at the grade four and six levels indicate the IFS could profitably be incorporated at a higher grade level, perhaps beginning in grades five and six.

Ginther et al. (1977) in a survey on student achievement found that students in grade eight have a reasonable understanding of the fraction concept but not of the structure of rational numbers. They suggest that much instruction on fractions should be left until junior high. Results from the grade eight fraction tests in the present study agree with the Ginther findings.

Suggestions for Further Research

On the basis of results from this study, several research projects should be carried out. Perhaps the most direct implication is for an experiment in which measure instruction is the major aspect of the experimental treatment. Instruction would focus on linear and area measure tasks similar to those included on the TBMC. Subdivision and unit iteration would be the primary concepts included prior to instruction on fraction concepts. A control group would receive fraction instruction only. Such a study would be most productively carried out at the four to six grade level.

Weekly observations recorded during this study indicated a wide range in the use of measurement models at the grade six level. Another study would involve observation of regular mathematics classrooms on a daily basis for the duration of the rational number unit. This would be carried out at the grades five and six level, the object being to categorize instruction by model use. Another dimension could involve student/teacher interaction. A relationship between measure ability and fraction test performance has been established. It is reasonable to hypothesize that instruction utilizing measurement models will result in higher fraction achievement than instruction that does not utilize measurement models. Such a hypothesis can be tested through observation of a large number of randomly selected classrooms.

The TBMC was designed, for this study, to establish the relationship between measure concepts and fraction concepts. It also provides a level of measure development found in grades four, six and eight students. The TBMC was adequate for the purposes of this study but modification of the test is desirable before it is put to further use. Redevelopment of the TBMC will involve several steps. First, items which have been established as belonging to the same subtest should be analyzed and additions or deletions made so that every aspect of the concept is included in the subtest. For example, linear measure items could include more variations in path configuration. A desirable end would establish a hierarchy of linear measure items requiring mastery at all levels for linear measure to be operational. Subtest items will be revised on the basis of item analysis and factor analysis at various age levels.

Items which do not cluster into neat subtests on the basis of factor analysis will be analyzed to determine whether the items are assessing a distinct concept not already included in a subtest. If the items load on unique factors it is possible that many other variations of the items should be considered. For example, where four items have been included to test a concept, they may be testing only extreme instances of the concept and ignoring other points on a continuum. The TBMC could be split into two batteries, linear and area, each composed of a hierarchy of subtests. A program of research and redevelopment of the TBMC should be guided by fundamental test considerations such as those put forth in *Standards for Educational and Psychological Tests (1974).*

The TBMC was designed to test measure concepts in the group classroom setting. The advantages of clinical testing and the interview situation are not available in such a context. It is probable that students perform at a lower level in group situations. Research should therefore be carried out on the relationship between performance on clinical tasks and performance on paper and pencil tasks in group testing situations. It was pointed out in Chapter IV that the TBMC was developed to provide a standard in keeping with existing classroom procedures. It is realistic to assess student development in terms of that which is expected in the classroom context and not an artificial, if desirable, one to one situation. The objective is to develop a group test which provides as precise a measure of ability as possible.

Concluding Remarks

The two purposes of this study were: to determine the relationship between measure concepts and rational number concepts and to determine the level of development of students at three grade levels on measure tasks. A test of basal measurement concepts, the TBMC, was developed to assess student measure skills. Results of a correlational analysis of the TBMC subtests with fraction test subtests provides the needed relationships at the three grade levels. Results of the canonical correlation analysis indicate a relational consistency between behavioral variables and the theoretical variables discussed in Chapter II. Results of statistical analyses of the TBMC indicate a sequential development of measure concepts in students from grades four through eight. Measure development as assessed by the TBMC lags behind developmental levels suggested by Piaget and others based on clinical assessment.

The TBMC is a potentially useful indicator of readiness for measure related learning, for example rational number concepts. Difficulties with the TBMC must be researched and subtests modified and expanded on the basis of further item and factor analyses. Usefulness of the test should be assessed in individual and group diagnostic situations.

On the basis of results obtained from this study, implications can be drawn for curriculum development and instruction. Use of measurement models should be accompanied by assessment of student understanding of the underlying measure properties. Where students

have not mastered the associated measure tasks, instruction should take place on measure objectives. Serious reservations are associated with the present inclusion of number line models in the curriculum. Evidence from this study suggests that students at the grade four level show an overwhelming inability to deal with both number line specific and general linear measure tasks. At the grade six level the implication is for training for transfer from the linear concepts, which are understood, to number line concepts which are not. At all levels of study, the relationship between measure and fraction behaviors suggest the need for transfer based instruction.

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APPENDIX A

Protocol for Administering the TBMC

Grade Four Instructions

1. Distribute tests. Ask students to complete the first page including name, grade, school, teacher, date, birthdate, age.

2. Beginning with question one, read every question aloud while students follow silently. Special attention is given to questions as described below.

3. Question 1: "paths" are described as paths along which an ant might walk. Two simple paths are drawn on the board:  Students are asked which path is longer.

4. Question 5: the idea of twice as much is explained by showing a piece of string 6 inches long and then a piece of string that is 12 inches long.

5. Question 8: the notion of equal parts is explained by drawing two circles of unequal size and comparing the halves of each. Other comparisons involve halves and thirds of the same circle.

6. Questions 10 and 11: twice as big is explained to mean twice as much room to play.

7. Question 11: the notion of "like" is explained by illustrations involving similar figures. For example, a small and large circle are alike. An equilateral triangle (drawn on the board) is "like" another equilateral triangle but it is not "like" an isosceles triangle. A square is not "like" a rectangle.

8. Question 12: the notion of being "equal to" is distinguished from being "congruent to" with illustrations involving rectangles which are equal (have the same number of unit subdivisions) but do not look the same (are not congruent). All illustrations are done on the blackboard.

Grade Six Instructions

Numbers 1, 2, 3, 5, 6, 7, and 8 are similar to the grade four instructions given above.

4. Question 5: "twice as much" is described by verbal examples such as having a candy bar. How much would you have if you had twice as much? How long would your desk be if it were twice as long?

9. Question 12: stress is placed on thinking of the regions as cakes, not fractions. Students are urged to answer the question as it is asked and not to impose numbers on the parts.

Grade Eight Instructions

The instructions are similar to those for grade six but in questions 10 and 11 the word "area" is used to explain what is being doubled. Grade eight students are told the figures in question 11 must be *similar* when the area is doubled. Emphasis is also placed on question 12. Do not interpret the shaded parts as fractions. If you do, your answers will not always be the same as when you answer the question asked.

TBMC

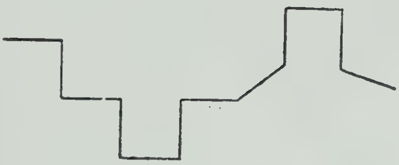
SCHOOL______________________________ GRADE____________________

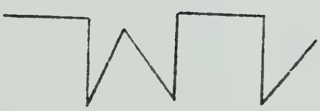
NAME________________________________ BOY_________ GIRL_________

BIRTHDATE____________________________ AGE______________________

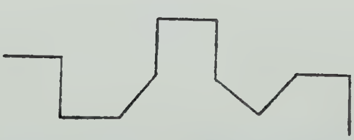
1. BELOW ARE TWO PATHS, A AND B. IS PATH A LONGER THAN B, SHORTER THAN B, OR ARE THEY THE SAME? (Answer on dotted line).

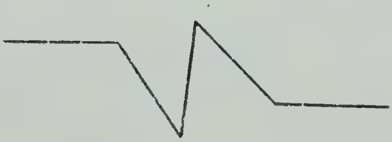
A


B


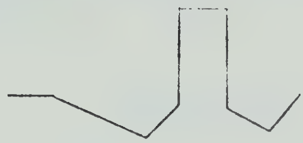
A


B


A


B


A



B

A

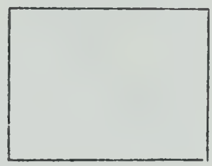
B

2. DIVIDE THE LINE SEGMENT BELOW INTO 6 EQUAL PARTS.

3. DIVIDE THE LINE SEGMENT BELOW INTO 5 EQUAL PARTS.

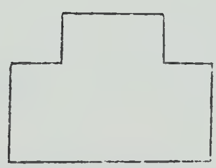
4. BELOW ARE TWO YARDS, A AND B, WITH FENCES AROUND THEM. COULD MORE PEOPLE FIT INTO A OR B OR SHOULD THEY HOLD THE SAME AMOUNT?

A



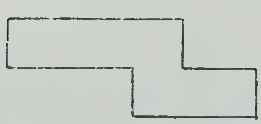
B

A

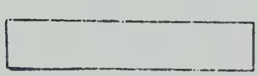


B

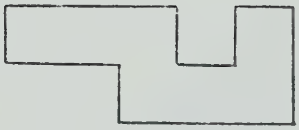
A



B



A

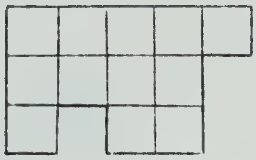


B

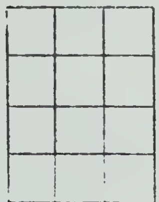
5. DRAW A LINE THAT IS TWICE AS LONG AS THE ONE SHOWN BELOW.

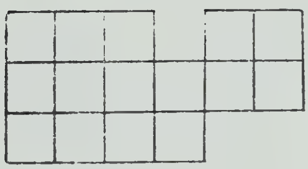
6. DRAW A LINE THAT IS 5 TIMES AS LONG AS THE ONE SHOWN BELOW.

7. BELOW ARE TWO ROOMS, A AND B. THE ROOMS HAVE TILE ON THE FLOOR. WOULD YOU HAVE MORE SPACE TO PLAY IN ROOM A OR B OR ARE THEY THE SAME?

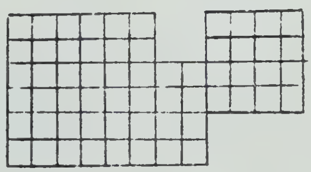
A


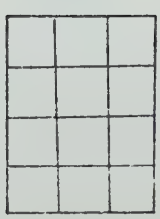
- - - - - - - - - - - - - - -

B


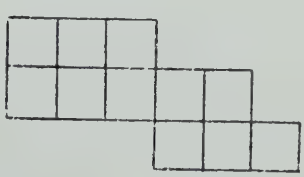
A


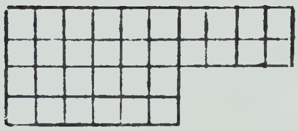
- - - - - - - - - - - - -

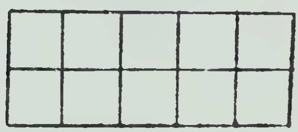
B


A


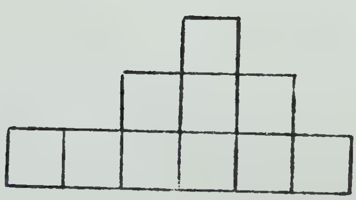
- - - - - - - - - - - - -

B


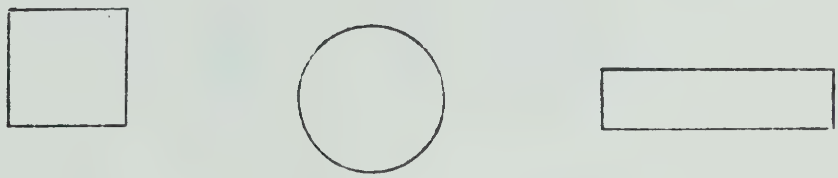
A 

B 

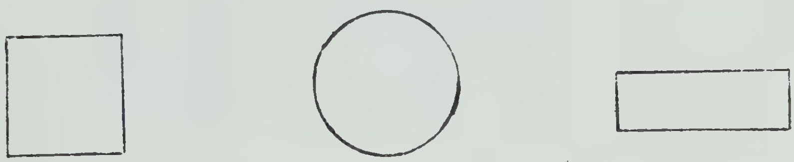
A 

B 

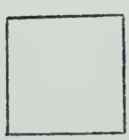
8. SHOW HOW YOU WOULD DIVIDE EACH FIGURE BELOW TO GET 5 EQUAL PARTS.



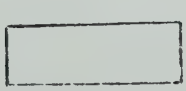
9. SHOW HOW YOU WOULD DIVIDE EACH FIGURE BELOW TO GET 6 EQUAL PARTS.



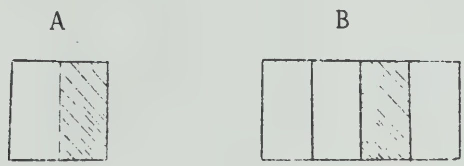
10. BELOW IS A SQUARE FIGURE. DRAW ANOTHER SQUARE WHICH IS TWICE AS BIG AS THE ONE SHOWN.

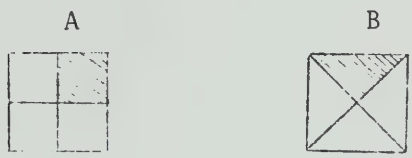


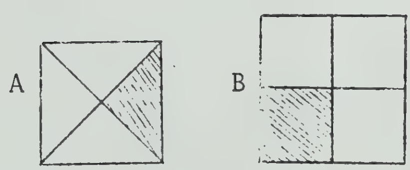
11. DRAW A RECTANGLE LIKE THE ONE SHOWN BUT MAKE IT TWICE AS BIG.

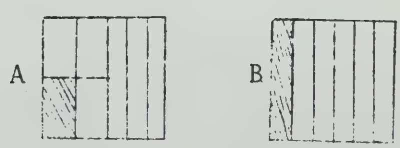


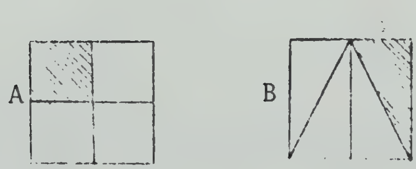
12. LOOK AT THE PAIRS OF FIGURES BELOW. THINK OF EACH FIGURE AS A CAKE. IN EACH CAKE TELL WHICH SHADED PIECE WOULD BE BIGGER, A OR B, OR ARE THEY THE SAME?









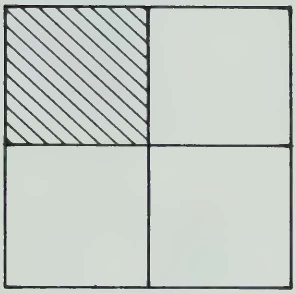


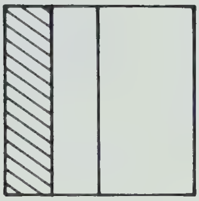
Grade Four

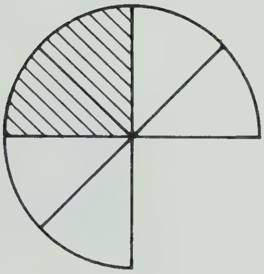
Fraction Achievement Test*

*Figures in parentheses represent factor analysis equivalents.

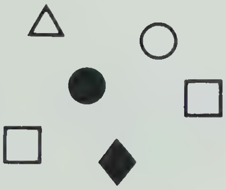
1. For each picture write a fraction that tells what part is shaded.

A.)  = ______ (1)

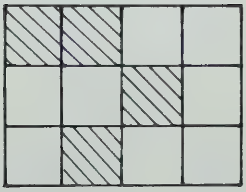
B.)  = ______ (2)

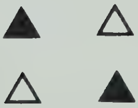
C.)  = ______ (3)

D.)  = ______ (4)

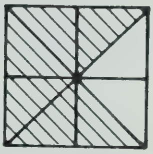
E.)  = ______ (5)

F.)  = ______ (6)

G.)  = ______ (7)

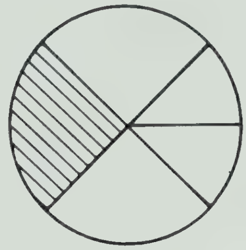
H.)  = ______ (8)

2. Write <u>two</u> fractions to tell what part is shaded.

A.) 

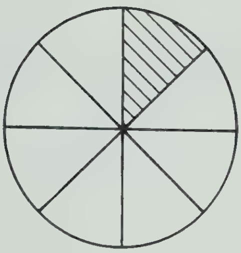
= ______ (9)

= ______ (10)

B.) 

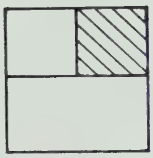
= ______ (11)

= ______ (12)

C.) 

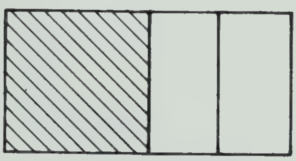
= ______ (13)

= ______ (14)

D.) 

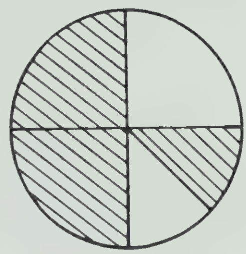
= ______ (15)

= ______ (16)

E.) 

= ______ (17)

= ______ (18)

F.) 

= ______ (19)

= ______ (20)

3. Write the missing denominators to get a set of equivalent fractions.

A.) $\frac{1}{2}, \frac{2}{4}, \frac{3}{\square}$ (21)

B.) $\frac{3}{5}, \frac{6}{\square}, \frac{9}{15}$ (22)

C.) $\frac{1}{7}, \frac{2}{14}, \frac{3}{\square}$ (23)

D.) $\frac{2}{3}, \frac{4}{\square}, \frac{6}{\square}, \frac{8}{12}$ (24) (25)

4. Write two more fractions for each set below to get equivalent sets.

A.) $\frac{1}{3}$, $\frac{2}{6}$, — (26), — (27)

B.) $\frac{3}{5}$, $\frac{6}{10}$, — (28), — (29)

C.) $\frac{7}{2}$, $\frac{14}{4}$, — (30), — (31)

D.) $\frac{11}{5}$, — (32), — (33)

5.* Write each fraction in lowest terms.

A.) $\frac{2}{4}$ = —

B.) $\frac{3}{21}$ = —

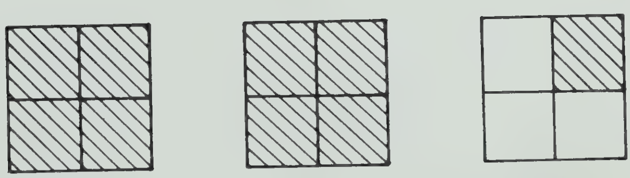
C.) $\frac{12}{15}$ = —

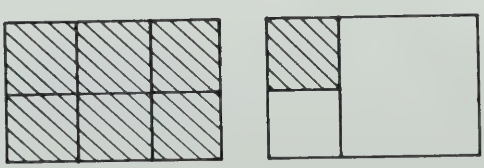
D.) $\frac{7}{9}$ =

E.) $\frac{1}{14}$ = —

F.) $\frac{15}{18}$ = —

6. Write an improper fraction for the shaded part of each picture.

A.)  = ——— (34)

B.)  = ——— (35)

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*Deleted from statistical analyses.

7. Place an <u>X</u> on the point of the number line named by the fraction.

$\frac{3}{4}$ 

(36)

$\frac{2}{5}$ 

(37)

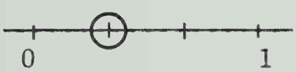
$\frac{2}{3}$ 

(38)

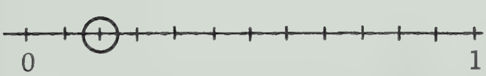
$\frac{7}{8}$ 

(39)

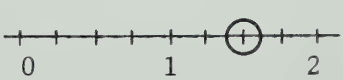
8. Write the fraction name for each point circled below.



= _____ (40)



= _____ (41)



= _____ (42)

= _____ (43)

9.* Write the correct sign: < , > , or = .

A.) $\frac{1}{2} \square \frac{1}{3}$ B.) $\frac{1}{2} \square \frac{3}{13}$ C.) $\frac{2}{5} \square \frac{3}{4}$

D.) $\frac{2}{2} \square 2$ E.) $3 \square \frac{1}{2}$ F.) $\frac{5}{6} \square 5$

G.) $\frac{4}{5} \square \frac{11}{12}$ H.) $3\frac{1}{2} \square \frac{21}{7}$ I.) $2\frac{1}{3} \square 8\frac{1}{3}$

10* Find the sums.

A.) $\frac{1}{3} + \frac{1}{3} =$ ____ B.) $\frac{1}{5} + \frac{2}{5} =$ ____

C.) $\frac{1}{7} + \frac{6}{7} =$ ____ D.) $\frac{1}{2} + \frac{2}{4} =$ ____

E.) $\frac{1}{3} + \frac{1}{2} =$ ____ F.) $\frac{1}{6} + 3 =$ ____

*Deleted from statistical analyses.

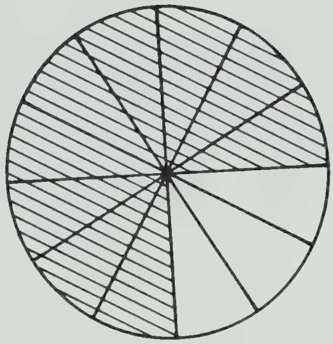
Grade Six

Fraction Achievement Test*

*Figures in parentheses represent factor analysis equivalents.

1. a.) Give a fraction that tells what part of the circular area is shaded.

_________ (1)

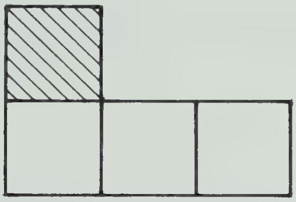


b.) Give a different fraction for the shaded area. _________ (2)

c.) Give a fraction that tells what part of the circular area is not shaded.

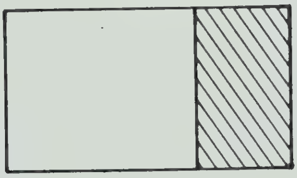
_________ (3)

d.) What fractional part of the figure below is shaded?



_________ (4)

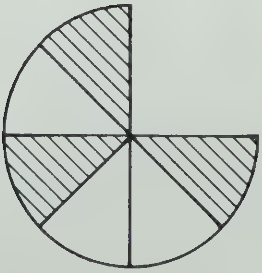
e.) What fractional part of the figure below is shaded?



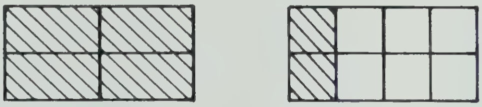
_________ (5)

f.) What fractional part of the figure below is shaded?

_________ (6)

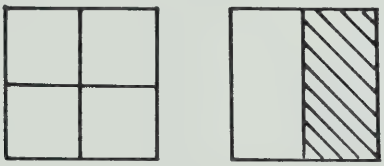


g.) Name the mixed number that is represented by the shaded region below.



__________ (7)

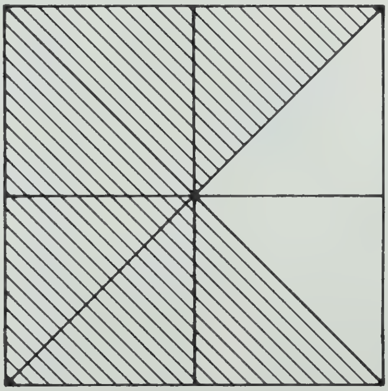
h.)* Name the mixed number that is represented by the <u>un</u>shaded region below.



____

2. a.) What fraction tells you how much of the square region is shaded?

__________ (8)



b.) Give another fraction for the shaded region.

__________ (9)

c.) Give a fraction for the part that is <u>not</u> shaded.

__________ (10)

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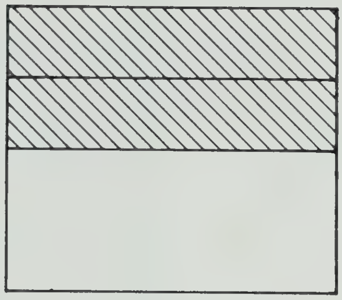
*Deleted from analyses.

3. a.) Give a fraction that tells how much of the square region is shaded.

_________ (11)

b.) Give <u>two</u> fractions that tell how much of the square is <u>not</u> shaded.

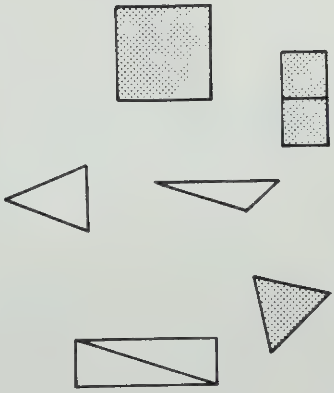


_________ (12) _________ (13)

4. a.) Give a fraction that tells what part of the set of figures is shaded.

_________ (14)

b.) What fractional part of the set is <u>not</u> shaded?



_________ (15)

c.) What fractional part of the set are squares?

_________ (16)

d.)* What fractional part of the set are triangles?

____

*Deleted from analyses.

5. Place each fraction below on the number line to the right.

$\frac{1}{3}$ 

(17)

$\frac{2}{5}$ 

(18)

$\frac{5}{8}$ 

(19)

$\frac{2}{3}$ 

(20)

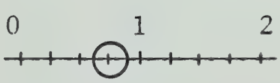
$\frac{4}{6}$ 

(21)

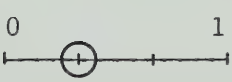
$\frac{3}{4}$ 

(22)

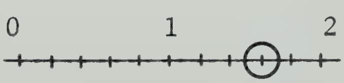
6. Give <u>two</u> fraction names for the point circled on each number line below.



________ (23) ________ (24)



________ (25) ________ (26)



________ (27) ________ (28)

7. a.) Name any <u>two</u> fractions that are equivalent to $\frac{1}{3}$.

$\frac{1}{3}$ = ____ (29) $\frac{1}{3}$ = ____ (30)

b.) Name any <u>two</u> fractions that are equivalent to $\frac{6}{10}$.

$\frac{6}{10}$ = ____ (31) $\frac{6}{10}$ = ____ (32)

c.) Name any <u>two</u> fractions that are equivalent to $\frac{21}{33}$.

$\frac{21}{33}$ = ____ (33) $\frac{21}{33}$ = ____ (34)

8. Give the lowest terms fraction for each fraction below.

a.) $\frac{20}{25}$ = ____ (35) b.) $\frac{16}{24}$ = ____ (36) c.) $\frac{4}{28}$ = ____ (37)

d.) $\frac{3}{13}$ = ____ (38) e.) $\frac{3}{6}$ = ____ (39)

9. Write each mixed numeral as an improper fraction.

a.) $2\frac{1}{4}$ = ____ (40) b.) $12\frac{4}{5}$ = ____ (41)

c.) $2\frac{3}{13}$ = ____ (42) d.) $3\frac{1}{11}$ = ____ (43)

10. Write each improper fraction as a mixed numeral or whole number.

a.) $\frac{21}{3}$ = ____ (44)   b.) $\frac{22}{8}$ = ____ (45)

c.) $\frac{38}{20}$ = ____ (46)   d.) $\frac{14}{5}$ = ____ (47)

11. Tell whether the first fraction in each pair is equivalent to, greater than, or less than the second. You may use the symbols $=$, $>$, $<$.

a.) $\frac{1}{3}$ □ $\frac{1}{4}$ (48)   b.) $\frac{5}{9}$ □ $\frac{4}{5}$ (49)   c.) $\frac{2}{3}$ □ $\frac{3}{5}$ (50)

d.) $\frac{2}{2}$ □ $\frac{9}{9}$ (51)   e.) $\frac{4}{3}$ □ $\frac{5}{4}$ (52)   f.) $\frac{21}{20}$ □ $\frac{42}{40}$ (53)

12. Find the <u>sums</u> <u>or</u> <u>differences</u> for each problem.

a.) $\frac{3}{6} + \frac{2}{6}$ = ____ (54)   b.) $\frac{1}{18} + \frac{3}{18}$ = ____ (55)   c.) $\frac{20}{30} + \frac{10}{30}$ = ____ (56)

d.) $\frac{5}{10} - \frac{2}{10}$ = ____ (57)   c.) $\frac{13}{13} - \frac{2}{13}$ = ____ (58)   d.) $\frac{27}{72} - \frac{10}{72}$ = ____ (59)

13. Find the <u>sums</u> <u>or</u> <u>differences</u> for each problem.

a.) $\frac{2}{8} + \frac{2}{4}$ = ____ (60)   b.) $\frac{3}{16} + \frac{1}{2}$ = ____ (61)   c.) $\frac{4}{5} + \frac{8}{7}$ = ____ (62)

d.) $\frac{3}{5} - \frac{1}{7}$ = ____ (63)   e.) $\frac{5}{6} - \frac{1}{3}$ = ____ (64)   f.) $\frac{7}{8} - \frac{1}{4}$ = ____ (65)

14. Find the sums.

a.) $\frac{1}{3} + \frac{1}{6} + \frac{1}{2} =$ ______ (66) b.) $\frac{2}{5} + \frac{1}{10} + \frac{6}{20} =$ ______ (67)

c.) $3\frac{1}{8} + 5\frac{1}{4} =$ ______ (68) d.) $14\frac{1}{2} + 2\frac{1}{4} =$ ______ (69)

15. Find the differences.

a.) $10\frac{1}{8} - 2\frac{1}{8} =$ ______ (70) b.) $3\frac{1}{3} - 2\frac{1}{12} =$ ______ (71)

16. Find the products.

a.) $\frac{1}{8} \times \frac{2}{5} =$ ______ (72) b.) $\frac{2}{7} \times \frac{1}{3} =$ ______ (73) c.) $9 \times \frac{3}{8} =$ ______ (74)

d.) $3\frac{2}{5} \times \frac{1}{2} =$ ______ (75) e.) $2\frac{2}{5} \times 1\frac{1}{8} =$ ______ (76) f.) $\frac{1}{6} \times \frac{3}{6} =$ ______ (77)

17. Find the quotients.

a.) $\frac{2}{5} \div \frac{2}{5} =$ ______ (78) b.) $3 \div \frac{1}{3} =$ ______ (79) c.) $\frac{1}{7} \div \frac{7}{11} =$ ______ (80)

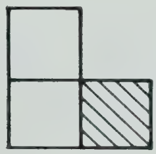
d.) $2\frac{2}{3} \div \frac{1}{2} =$ ______ (81) e.) $\frac{8}{5} \div \frac{1}{9} =$ ______ (82) f.) $\frac{1}{2} \div 3\frac{1}{3} =$ ______ (83)

Grade Eight

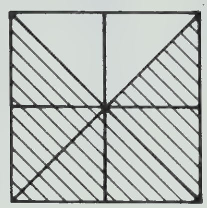
Fraction Achievement Test*

*Figures in parentheses represent factor analysis equivalents.

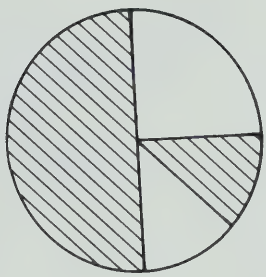
1. For each figure below give a fraction that tells what part is shaded.



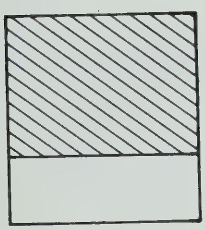
= ______ (1)



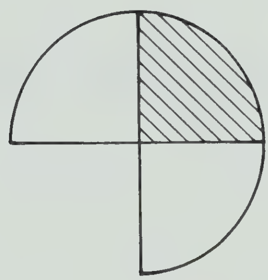
= ______ (2)



= ______ (3)

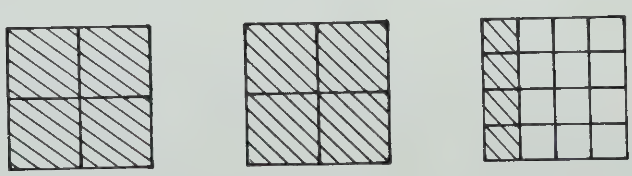


= ______ (4)

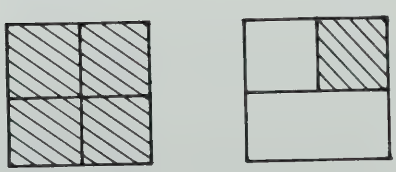


= ______ (5)

2. Name the mixed number that is represented by the shaded regions below.



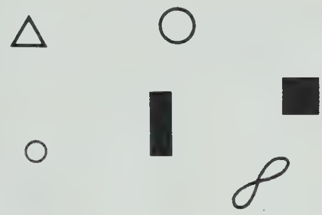
= ______ (6)



= ______ (7)

3. Give a fraction that tells what part of the set of figures is shaded.

= ______ (8)



What fractional part is <u>not</u> shaded?

= ______ (9)

What fractional part are squares?

= ______ (10)

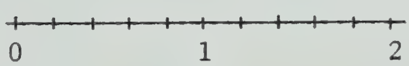
4. Place each fraction below on the number line to the right.

$\frac{1}{3}$ 

(11)

$\frac{2}{6}$ 

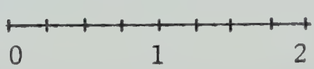
(12)

$\frac{1}{5}$ 

(13)

$\frac{6}{5}$ 

(14)

$\frac{5}{8}$ 

(15)

$\frac{3}{4}$ 

(16)

$\frac{1}{4}$ 

(17)

5. Give a fraction name for the point circled on each number line below.

= ______ (18)

= ______ (19)

= ______ (20)

6. Reduce each fraction to <u>lowest</u> terms.

$\frac{40}{50}$ = ______ (21) $\frac{6}{12}$ = ______ (22) $\frac{7}{13}$ = ______ (23) $\frac{3}{27}$ = ______ (24)

7. Change to improper fractions.

$2\frac{2}{13}$ = ______ (25) $3\frac{7}{11}$ = ______ (26) $12\frac{3}{5}$ = ______ (27)

8. Change to mixed numerals.

$\frac{21}{4}$ = ______ (28) $\frac{37}{20}$ = ______ (29) $\frac{36}{8}$ = ______ (30)

9. Name any <u>two</u> fractions that are equivalent to each fraction below.

$\frac{12}{30}$ = ______ (31) = ______ (32) $\frac{22}{33}$ = ______ (33) = ______ (34)

$\frac{1}{3}$ = ______ (35) = ______ (36)

10. Is the first fraction greater than (>), less than (<), or equal to (=) the second fraction for each pair?

$\frac{22}{40} \square \frac{11}{20}$ (37) $\frac{1}{3} \square \frac{1}{2}$ (38) $\frac{2}{11} \square \frac{3}{17}$ (39)

$\frac{4}{5} \square \frac{2}{3}$ (40) $\frac{6}{5} \square \frac{4}{3}$ (41)

11. Find the sums.

$\frac{1}{2} + \frac{2}{3} =$ ____ (42) $\frac{3}{5} + \frac{4}{7} =$ ____ (43) $\frac{1}{3} + \frac{2}{5} + \frac{1}{4} =$ ____ (44)

$\frac{1}{18} + \frac{5}{18} =$ ____ (45) $\frac{2}{7} + \frac{1}{7} =$ ____ (46) $2\frac{1}{8} + 3\frac{3}{8} =$ ____ (47)

$3\frac{2}{5} + 2\frac{1}{3} =$ ____ (48) $5\frac{1}{5} + 2\frac{1}{2} =$ ____ (49)

12. Find the differences.

$\frac{11}{17} - \frac{2}{17} =$ ____ (50) $\frac{4}{7} - \frac{2}{7} =$ ____ (51) $3\frac{1}{11} - \frac{2}{11} =$ ____ (52)

$3\frac{3}{8} - \frac{1}{4} =$ ____ (53) $5\frac{1}{7} - 1\frac{2}{7} =$ ____ (54) $4\frac{3}{4} - 2\frac{1}{5} =$ ____ (55)

13. Find the products.

$\frac{2}{3} \times \frac{1}{8} =$ ____ (56) $\frac{1}{2} \times \frac{1}{2} =$ ____ (57) $3\frac{1}{7} \times 2 =$ ____ (58)

$2\frac{2}{5} \times \frac{1}{5} =$ ____ (59) $4\frac{1}{2} \times \frac{3}{4} =$ ____ (60) $2\frac{2}{7} \times 3\frac{1}{5} =$ ____ (61)

14. Find the quotients.

$\frac{1}{4} \div \frac{1}{4} =$ ____ (62) $\frac{1}{5} \div 5 =$ ____ (63) $\frac{3}{4} \div \frac{1}{7} =$ ____ (64)

$\frac{5}{8} \div \frac{3}{7} =$ ____ (65) $2\frac{1}{7} \div \frac{2}{3} =$ ____ (66) $3\frac{2}{5} \div 2\frac{1}{5} =$ ____ (67)

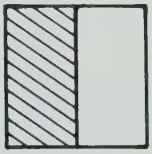
$3\frac{1}{4} \div 1\frac{1}{2} =$ ____ (68)

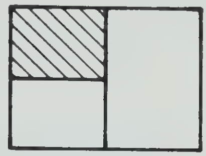
Grade Four

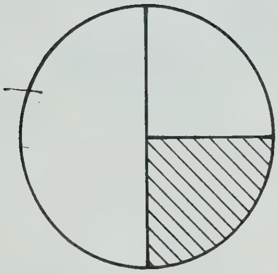
Fraction Retention Test*

*Figures in parentheses represent factor analysis equivalents.

1. For each picture write a fraction that tells what part is shaded.

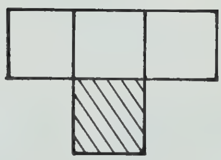
A.)  = ____ (1)

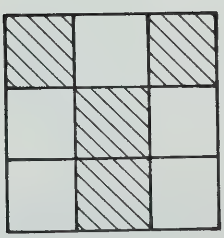
B.)  = ____ (2)

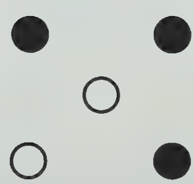
C.)  = ____ (3)

D.)  = ____ (4)

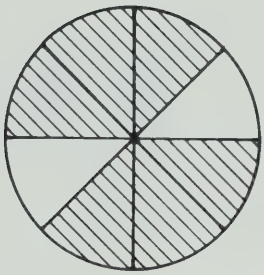
E.)  = ____ (5)

F.)  = ____ (6)

G.)  = ____ (7)

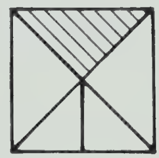
H.)  = ____ (8)

2. Write two fractions to tell what part is shaded.

A.) 

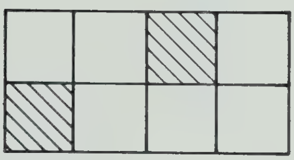
= ____ (9)

= ____ (10)

B.) 

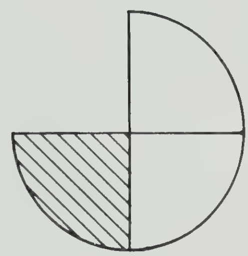
= ____ (11)

= ____ (12)

C.) 

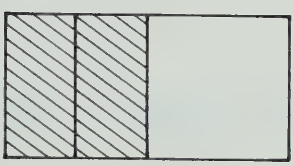
= ____ (13)

= ____ (14)

D.) 

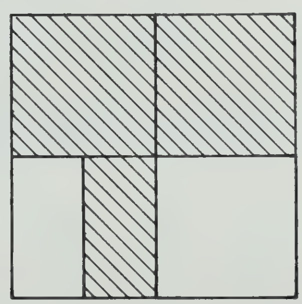
= ____ (15)

= ____ (16)

E.) 

= ____ (17)

= ____ (18

F.) 

= ____ (19)

= ____ (20)

3. Write the missing denominator to get a set of equivalent fractions.

A.) $\frac{1}{2}$, $\frac{2}{\square}$, $\frac{3}{6}$ (21)

B.) $\frac{3}{5}$, $\frac{6}{10}$, $\frac{9}{\square}$ (22)

C.) $\frac{1}{7}$, $\frac{2}{\square}$, $\frac{3}{21}$ (23)

D.) $\frac{2}{3}$, $\frac{4}{6}$, $\frac{6}{\square}$, $\frac{8}{\square}$ (24) (25)

4. Write two more fractions for each set below to get equivalent sets.

A.) $\frac{1}{2}$, $\frac{2}{4}$, — (26), — (27) B.) $\frac{2}{5}$, $\frac{4}{10}$, — (28), — (29)

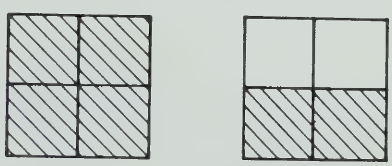
C.) $\frac{3}{7}$, $\frac{6}{14}$, — (30), — (31) D.) $\frac{11}{7}$, — (32), — (33)

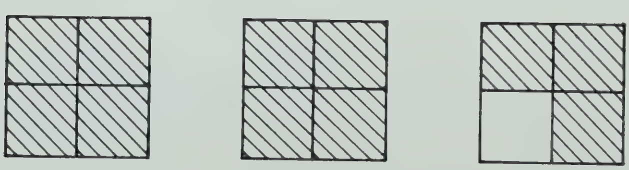
5.* Write each fraction in lowest terms.

A.) $\frac{3}{6}$ = ____ B.) $\frac{4}{20}$ = ____ C.) $\frac{12}{15}$ = ____

D.) $\frac{5}{7}$ = ____ E.) $\frac{6}{16}$ = ____ F.) $\frac{15}{18}$ = ____

6. Write an improper fraction for the shaded part of each picture.

A.)  = ____ (34)

B.)  = ____ (35)

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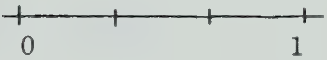
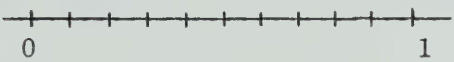
*Deleted from statistical analyses.

7. Place an X on the point of the number line named by the fraction.

$\frac{1}{3}$ 

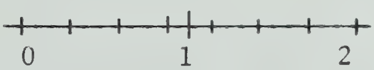
(36)

$\frac{3}{5}$ 

(37)

$\frac{3}{2}$ 

(38)

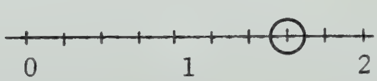
$\frac{5}{7}$ 

(39)

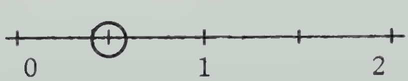
8. Write the fraction name for each point circled below.

= ____ (40)

= ____ (41)



= ____ (42)



= ____ (43)

9.* Write the correct sign: $<$, $>$, or $=$.

A.) $\frac{1}{3} \square \frac{1}{2}$ B.) $\frac{3}{13} \square \frac{1}{2}$ C.) $\frac{1}{2} \square 3$

D.) $\frac{11}{12} \square \frac{4}{5}$ E.) $\frac{21}{7} \square 3\frac{1}{2}$ F.) $2\frac{1}{3} \square 8\frac{1}{2}$

10.* Find the sums.

A.) $\frac{1}{2} + \frac{1}{2} =$ ______ B.) $\frac{3}{5} + \frac{1}{5} =$ ______

C.) $\frac{2}{8} + \frac{5}{8} =$ ______ D.) $\frac{2}{4} + \frac{1}{2} =$ ______

E.) $\frac{1}{2} + \frac{1}{3} =$ ______ F.) $\frac{1}{7} + 2 =$ ______

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*Deleted from statistical analyses.

Grade Six

Fraction Retention Test*

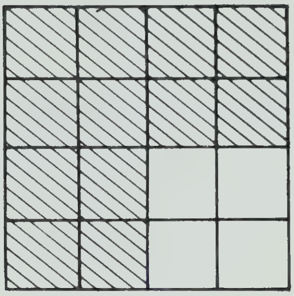
*Figures in parentheses represent factor analysis equivalents.

1. a.) Give a fraction that tells what part of the region is shaded.

= ________ (1)

b.) Give a different fraction for the shaded area.

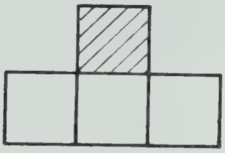
= ________ (2)



c.) Give a fraction that tells what part is not shaded.

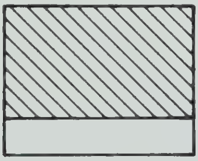
= ________ (3)

d.) What fractional part of the figure below is shaded?



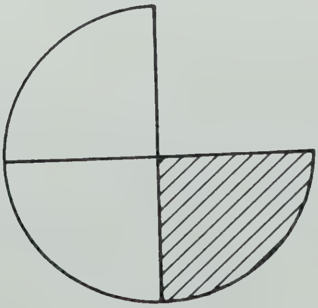
= ________ (4)

e.) What fractional part of the figure below is shaded?



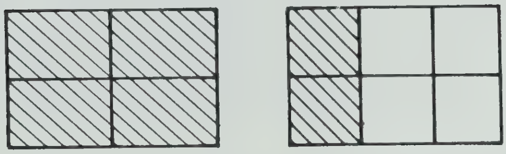
= ________ (5)

f.) What fractional part of the figure below is shaded?



= ________ (6)

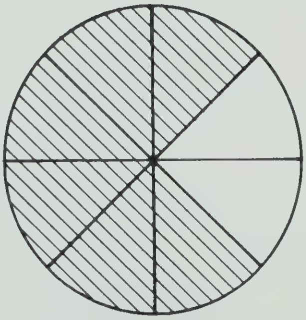
g.) Name the mixed number that is represented by the shaded region below.



= ________ (7)

2. a.) What fraction tells you how much of the circular area is shaded?

= ________ (8)



b.) Give another fraction for the shaded area.

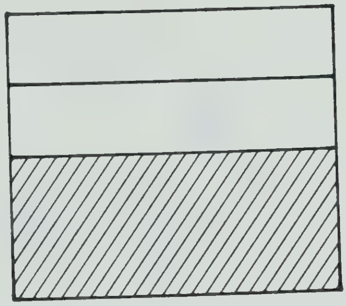
= ________ (9)

c.) Give a fraction for the part of the circle that is not shaded.

= ________ (10)

3. a.) Give a fraction that tells how much of the region is shaded.

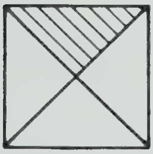
= ________ (11)



b.) Give two fractions to tell what part is not shaded.

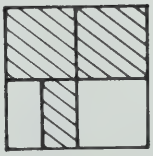
= ________ (12) = ________ (13)

c.) Give two fractions to tell what part of each figure below is shaded.



= ________ (14)

= ________ (15)

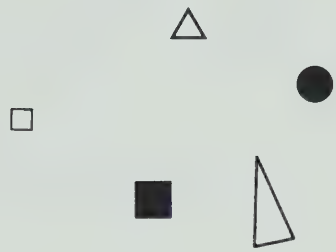


= ________ (16)

= ________ (17)

4. a.) Give a fraction that tells what part of the set of figures is shaded.

= ________ (18)



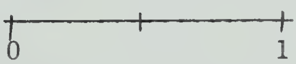
What fraction part is <u>not</u> shaded?

= ________ (19)

What fractional part of the set are squares?

= ________ (20)

5. Place each fraction below on the number line to the right.

$\frac{1}{4}$ 

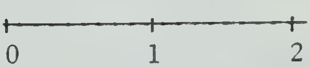
(21)

$\frac{3}{4}$ 

(22)

$\frac{7}{8}$ 

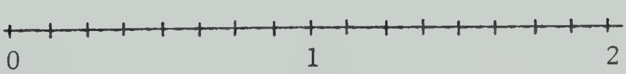
(23)

$\frac{1}{3}$ 

(24)

$\frac{2}{6}$ 

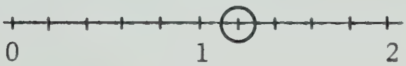
(25)

$\frac{5}{4}$ 

(26)

6. Give two fraction names for the point circled on each number line below.

= ______ (27) = ______ (28)



= ______ (29) = ______ (30)

= ______ (31) = ______ (32)

7. a.) Name any two fractions that are equivalent to $\frac{1}{4}$.

$\frac{1}{4}$ = ______ (33) $\frac{1}{4}$ = ______ (34)

b.) Name any two fractions that are equivalent to $\frac{8}{10}$.

$\frac{8}{10}$ = ______ (35) $\frac{8}{10}$ = ______ (36)

c.) Name any two fractions that are equivalent to $\frac{15}{33}$.

$\frac{15}{33}$ = ______ (37) $\frac{15}{33}$ = ______ (38)

8. Give the lowest terms fraction for each fraction below.

a.) $\frac{40}{50}$ = ——— (39) b.) $\frac{8}{12}$ = ——— (40) c.) $\frac{3}{27}$ = ——— (41)

d.) $\frac{7}{11}$ = ——— (42) e.) $\frac{6}{12}$ = ——— (43)

9. Write each mixed numeral as an improper fraction.

a.) $2\frac{1}{3}$ = ——— (44) b.) $12\frac{3}{5}$ = ——— (45)

c.) $2\frac{2}{13}$ = ——— (46) d.) $3\frac{2}{11}$ = ——— (47)

10. Write each improper fraction as a mixed numeral or whole number.

a.) $\frac{21}{4}$ = ——— (48) b.) $\frac{36}{8}$ = ——— (49)

c.) $\frac{37}{20}$ = ——— (50) d.) $\frac{13}{5}$ = ——— (51)

11. Tell whether the first fraction in each pair is equivalent to, greater than, or less than the second. You may use the symbols $=$, $>$, $<$.

a.) $\frac{1}{4}$ ☐ $\frac{1}{3}$ (52) b.) $\frac{4}{5}$ ☐ $\frac{5}{9}$ (53) c.) $\frac{3}{5}$ ☐ $\frac{2}{3}$ (54)

d.) $\frac{7}{7}$ ☐ $\frac{3}{3}$ (55) e.) $\frac{5}{4}$ ☐ $\frac{4}{3}$ (56) f.) $\frac{44}{40}$ ☐ $\frac{22}{20}$ (57)

12. Find the <u>sums</u> or <u>differences</u> for each problem.

a.) $\frac{3}{6} + \frac{1}{6} =$ ______ (58) $\frac{3}{18} + \frac{1}{18} =$ ______ (59) $\frac{11}{30} + \frac{7}{30} =$ ______ (60)

b.) $\frac{4}{10} - \frac{3}{10} =$ ______ (61) $\frac{11}{13} - \frac{3}{13} =$ ______ (62) $\frac{23}{72} - \frac{11}{72} =$ ______ (63)

13. Find the <u>sums</u> or <u>differences</u> for each problem.

a.) $\frac{1}{8} + \frac{2}{4} =$ ______ (64) $\frac{5}{16} + 8\frac{1}{2} =$ ______ (65) $\frac{3}{5} + \frac{8}{7} =$ ______ (66)

b.) $\frac{4}{5} - \frac{1}{7} =$ ______ (67) $\frac{3}{6} - \frac{1}{3} =$ ______ (68) $\frac{5}{8} - \frac{1}{4} =$ ______ (69)

14. Find the sums.

a.) $\frac{1}{6} + \frac{1}{2} + \frac{1}{3} =$ ______ (70) $\frac{3}{5} + \frac{1}{10} + \frac{4}{20} =$ ______ (71)

b.) $2\frac{1}{8} + 4\frac{1}{4} =$ ______ (72) $13\frac{1}{2} + 1\frac{1}{4} =$ ______ (73)

15. Find the differences.

a.) $10\frac{2}{8} - 3\frac{1}{8} =$ ______ (74) b.) $3\frac{1}{3} - 2\frac{1}{12} =$ ______ (75)

16. Find the products.

a.) $\frac{2}{8} \times \frac{2}{5} =$ ______ (76) $\frac{3}{7} \times \frac{2}{3} =$ ______ (77) $7 \times \frac{5}{8} =$ ______ (78)

b.) $2\frac{2}{5} \times \frac{1}{2} =$ ______ (79) $2\frac{3}{5} \times 1\frac{1}{8} =$ ______ (80) $\frac{1}{4} \times \frac{3}{4} =$ ______ (81)

17. Find the quotients.

a.) $\frac{3}{5} \div \frac{3}{5} =$ ______ (82) $5 \div \frac{1}{5} =$ ______ (83) $\frac{1}{6} \div \frac{6}{7} =$ ______ (84)

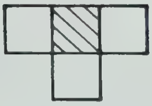
b.) $2\frac{2}{3} \div \frac{1}{2} =$ ______ (85) $\frac{7}{6} \div \frac{1}{3} =$ ______ (86) $\frac{1}{3} \div 3\frac{1}{2} =$ ______ (87)

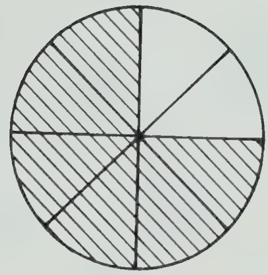
Grade Eight

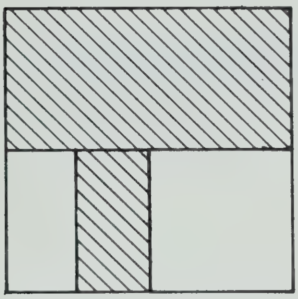
Fraction Retention Test*

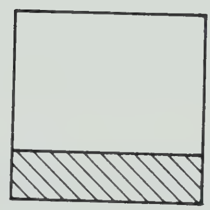
*Figures in parentheses represent factor analysis equivalents.

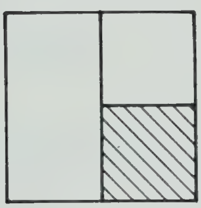
1. For each figure below give a fraction that tells what part is shaded.

*  = ______

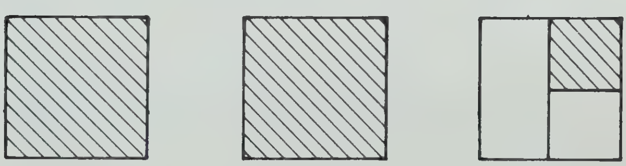
 = ______ (1)

 = ______ (2)

 = ______ (4)

 = ______ (3)

2. Name the improper fraction that is represented by the shaded region below.

 = ______ (5)

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*Deleted from factor analysis.

3. Give a fraction that tells what part of the set of figures is shaded.

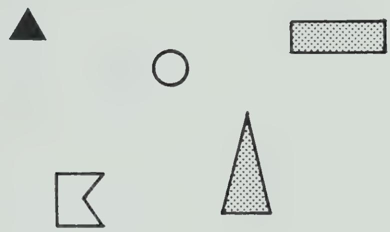
= ______ (6)

What fractional part is not shaded?

= ______ (7)

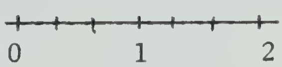
What fractional part are triangles?

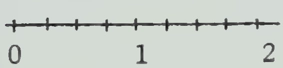
= ______ (8)



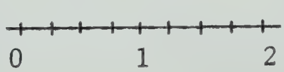
4. Place each fraction below on the number line to the right.

$\frac{2}{3}$  (9)

$\frac{5}{6}$  (10)

$\frac{1}{4}$  (11)

$\frac{7}{5}$  (12)

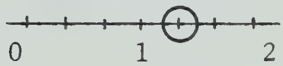
$\frac{3}{8}$  (13)

$\frac{5}{4}$  (14)

$\frac{1}{3}$  (15)

5. Give a fraction name for the point circled on each number line below.

= ____ (16)



= ____ (17)

= ____ (18)

6. Reduce to lowest terms.

$\frac{7}{21}$ = ____ (19) $\frac{20}{36}$ = ____ (20) $\frac{4}{8}$ = ____ (21) $\frac{6}{14}$ = ____ (22)

7. Change to improper fractions.

$3\frac{1}{8}$ = ____ (23) $2\frac{5}{13}$ = ____ (24) $10\frac{4}{5}$ = ____ (25)

8. Change to mixed numerals.

$\frac{33}{10}$ = ____ (26) $\frac{48}{20}$ = ____ (27) $\frac{17}{4}$ = ____ (28)

9. Name any <u>two</u> fractions that are equivalent to each fraction below.

$\frac{2}{3}$ = ____ (29) = ____ (30) $\frac{5}{10}$ = ____ (31) = ____ (32)

$\frac{21}{33}$ = ____ (33) = ____ (34)

10. Is the first fraction greater than (>), less than (<), or equal to (=) the second fraction for each pair?

$\frac{1}{4} \square \frac{1}{5}$ (35) $\frac{2}{10} \square \frac{21}{101}$ (36) $\frac{2}{13} \square \frac{4}{17}$ (37)

$\frac{3}{5} \square \frac{2}{3}$ (38) $\frac{5}{3} \square \frac{7}{5}$ (39)

11. Find the sums.

$\frac{3}{7} + \frac{2}{7} =$ ___ (40) $\frac{5}{18} + \frac{7}{18} =$ ___ (41) $\frac{1}{3} + \frac{3}{4} =$ ___ (42)

$\frac{1}{2} + \frac{2}{3} + \frac{1}{5} =$ ___ (43) $3\frac{1}{7} + 2\frac{4}{7} =$ ___ (44) $5\frac{1}{2} + 2\ \frac{3}{4} =$ ___ (45)

12. Find the differences.

$\frac{7}{12} - \frac{2}{12} =$ ___ (46) $\frac{3}{7} - \frac{2}{7} =$ ___ (47) $4\frac{1}{8} - \frac{3}{8} =$ ___ (48)

$5\frac{5}{6} - \frac{2}{3} =$ ___ (49) $2\frac{3}{4} - 2\frac{2}{5} =$ ___ (50)

13. Find the products.

$\frac{3}{5} \times \frac{1}{7} =$ ____ (51) $\frac{1}{3} \times \frac{1}{3} =$ ____ (52) $3\frac{1}{6} \times 2 =$ ____ (53)

$2\frac{2}{7} \times \frac{1}{7} =$ ____ (54) $5\frac{1}{2} \times \frac{1}{4} =$ ____ (55) $2\frac{1}{5} \times 7\frac{1}{7} =$ ____ (56)

14. Find the quotients.

$\frac{3}{4} \div \frac{3}{4} =$ ____ (57) $\frac{2}{7} \div 7 =$ ____ (58) $\frac{2}{5} \div \frac{1}{7} =$ ____ (59)

$3\frac{2}{5} \div \frac{3}{8} =$ ____ (60) $2\frac{3}{7} \div 3\frac{1}{7} =$ ____ (61) $3\frac{1}{8} \div 1\frac{1}{2} =$ ____ (62)

APPENDIX B

TABLE 60. ITEM ANALYSIS FOR TBMC GRADE FOUR

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
1	.71	.27	.11	.09	.13
2	.62	.28	.12	.11	.28
3	.57	.52	.36	.21	.49
4	.28	.43	.27	.14	.41
5	.35	.46	.30	.17	.41
6	.27	.66	.43	.20	.52
7	.30	.58	.43	.20	.52
8	.73	.65	.49	.21	.54
9	.30	.17	.02	.06	.19
10	.80	.28	.13	.08	.21
11	.62	.17	.01	.07	.17
12	.65	.47	.31	.17	.49
13	.48	.60	.44	.24	.60
14	.71	.65	.49	.22	.61
15	.61	.58	.42	.22	.57
16	.69	.58	.42	.20	.53
17	.56	.54	.38	.21	.57
18	.58	.55	.39	.22	.64
19	.15	.39	.25	.09	.22
20	.08	.65	.52	.10	.19
21	.24	.34	.19	.11	.30
22	.14	.36	.22	.08	.22
23	.07	.45	.35	.07	.11
24	.17	.13	.02	.03	.08
25	.65	.07	.09	.03	.13
26	.25	.43	.28	.14	.30
27	.55	.18	.35	.07	-.23

(Cont'd)

TABLE 60 (Cont'd)

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
28	.37	.21	.05	.08	.23
29	.52	.12	.28	.05	-.20
30	.39	.51	.35	.20	.56

$n = 109$ $S^2 = 14.95$ KR-20 = .61

$\overline{X} = 13.50$ $S = 3.87$ $S.E._{meas} = 2.40$

TABLE 61. ITEM ANALYSIS FOR TBMC GRADE SIX

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
1	.76	.38	.24	.12	.28
2	.86	.52	.40	.12	.24
3	.67	.38	.24	.14	.35
4	.47	.49	.34	.19	.51
5	.53	.58	.43	.23	.57
6	.68	.50	.36	.18	.45
7	.69	.52	.37	.18	.43
8	.80	.42	.29	.12	.28
9	.41	.08	.07	.03	.04
10	.88	.43	.31	.09	.17
11	.59	.59	.45	.23	.58
12	.85	.54	.41	.12	.24
13	.67	.38	.24	.14	.30
14	.83	.47	.34	.12	.34
15	.88	.53	.41	.11	.22
16	.80	.70	.57	.19	.49
17	.77	.42	.29	.13	.25
18	.80	.63	.50	.18	.44
19	.38	.55	.41	.21	.50
20	.21	.49	.36	.14	.36
21	.64	.49	.35	.19	.45
22	.43	.52	.38	.21	.55
23	.23	.42	.28	.13	.34
24	.45	.50	.35	.20	.56
25	.74	.09	.05	.03	.05
26	.47	.44	.29	.17	.49
27	.42	.25	.11	.10	.23

(Cont'd)

TABLE 61 (Cont'd)

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
28	.44	.29	.14	.11	.28
29	.47	.09	.06	.04	.08
30	.54	.50	.35	.20	.44

$n = 137$ $S^2 = 18.56$ KR-20 = .70

$\overline{X} = 18.37$ $S = 4.31$ $S.E._{meas} = 2.36$

TABLE 62. ITEM ANALYSIS FOR TBMC GRADE EIGHT

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
1	.86	.50	.38	.11	.32
2	.84	.32	.19	.08	.16
3	.74	.41	.27	.13	.32
4	.56	.41	.27	.16	.39
5	.62	.55	.42	.21	.56
6	.80	.52	.39	.15	.40
7	.70	.51	.38	.18	.48
8	.78	.50	.37	.15	.42
9	.51	.24	.10	.10	.24
10	.88	.39	.27	.08	.21
11	.70	.37	.24	.13	.27
12	.93	.39	.28	.06	.13
13	.81	.51	.39	.14	.40
14	.93	.70	.60	.09	.18
15	.90	.64	.52	.11	.29
16	.93	.61	.51	.09	.19
17	.89	.59	.48	.11	.21
18	.91	.77	.65	.12	.26
19	.39	.37	.24	.14	.38
20	.51	.66	.52	.26	.67
21	.69	.69	.55	.25	.64
22	.52	.43	.29	.17	.47
23	.52	.68	.55	.27	.67
24	.62	.62	.49	.24	.56
25	.78	.36	.23	.11	.25
26	.71	.67	.53	.23	.51
27	.56	.29	.15	.11	.33

(Cont'd)

TABLE 62 (Cont'd)

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
28	.60	.28	.14	.11	.23
29	.53	.43	.29	.17	.41
30	.69	.69	.56	.24	.53

N 146 S^2 = 20.22 KR-20 = .76

$\overline{X}$ = 21.38 S = 4.50 $S.E._{meas}$ = 2.22

TABLE 63. ITEM ANALYSIS FOR FRACTION ACHIEVEMENT TEST GRADE FOUR

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
1	.92	.97	.91	.15	.33
2	.53	.91	.83	.36	.93
3	.65	.13	.05	.05	.14
4	.62	.78	.71	.30	.71
5	.88	.72	.66	.14	.33
6	.74	.27	.20	.09	.24
7	.94	1.02	.96	.13	.26
8	.95	.83	.78	.08	.19
9	.84	.70	.63	.17	.44
10	.55	.88	.81	.35	.93
11	.55	.82	.75	.33	.86
12	.46	.73	.66	.29	.76
13	.82	.54	.47	.14	.34
14	.47	.57	.49	.23	.65
15	.57	.93	.86	.37	.93
16	.42	.86	.78	.34	.86
17	.63	.79	.72	.30	.78
18	.56	.83	.76	.33	.82
19	.33	.89	.81	.32	.86
20	.29	.89	.81	.31	.80
21	.95	.67	.62	.08	.22
22	.96	.78	.73	.06	.15
23	.91	.84	.78	.14	.30
24	.95	.83	.78	.09	.22
25	.88	.69	.62	.14	.34
26	.95	.72	.66	.07	.15
27	.95	.82	.77	.09	.19
28	.89	.67	.60	.13	.26
29	.89	.74	.67	.14	.30

(Cont'd)

TABLE 63 (Cont'd)

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
30	.88	.76	.69	.15	.30
31	.86	.64	.57	.14	.30
32	.64	.60	.53	.22	.60
33	.63	.61	.53	.23	.64
34	.26	.64	.56	.21	.53
35	.25	.67	.60	.21	.53
36	.80	.70	.63	.20	.44
37	.21	.48	.41	.14	.26
38	.56	.54	.46	.21	.50
39	.06	.77	.72	.09	.20
40	.52	.57	.49	.23	.55
41	.45	.71	.64	.28	.76
42	.08	.76	.70	.12	.30
43	.08	.79	.73	.12	.30

$n = 109$ $S^2 = 67.52$ KR-20 = .92

$\overline{X} = 27.33$ $S = 8.72$ $S.E._{meas} = 2.32$

TABLE 64. ITEM ANALYSIS FOR FRACTION ACHIEVEMENT TEST GRADE SIX

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
1	.82	.20	.17	.05	.17
2	.75	.37	.33	.12	.32
3	.85	.51	.47	.12	.28
4	.73	.63	.59	.21	.51
5	.88	.51	.48	.11	.26
6	.91	.57	.54	.10	.23
7	.88	.46	.43	.09	.20
8	.73	.34	.30	.11	.32
9	.82	.62	.59	.17	.41
10	.19	.62	.59	.17	.53
11	.26	.54	.50	.17	.51
12	.44	.55	.51	.22	.53
13	.73	.62	.58	.21	.48
14	.26	.66	.62	.22	.63
15	.91	.83	.80	.14	.26
16	.91	.84	.80	.14	.31
17	.85	.62	.59	.15	.30
18	.85	.88	.85	.21	.44
19	.80	.85	.82	.24	.56
20	.79	.56	.52	.16	.36
21	.87	.74	.71	.16	.31
22	.91	.85	.82	.14	.23
23	.85	.51	.48	.12	.19
24	.92	.96	.92	.14	.21
25	.80	.76	.72	.22	.46
26	.83	.62	.58	.16	.38
27	.68	.65	.61	.23	.58
28	.85	.86	.83	.20	.44
29	.79	.63	.59	.18	.34

(Cont'd)

TABLE 64 (Cont'd)

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliabiligy	Discriminating Power
30	.77	.74	.70	.22	.45
31	.69	.93	.89	.33	.82
32	.59	.67	.63	.26	.64
33	.67	.83	.79	.30	.77
34	.66	.74	.69	.27	.59
35	.66	.86	.82	.32	.77
36	.64	.71	.67	.27	.57
37	.79	.46	.42	.13	.30
38	.45	.67	.63	.26	.67
39	.64	.39	.35	.14	.36
40	.43	.45	.41	.18	.48
41	.58	.60	.56	.24	.66
42	.34	.60	.55	.22	.66
43	.58	.67	.63	.26	.71
44	.37	.60	.56	.23	.61
45	.72	.61	.57	.21	.48
46	.93	.09	.06	.01	.07
47	.83	.23	.19	.06	.11
48	.90	.53	.50	.10	.20
49	.91	.41	.37	.07	.21
50	.89	.38	.35	.07	.20
51	.72	.28	.24	.10	.26
52	.54	.27	.23	.11	.30
53	.47	.66	.62	.26	.72
54	.31	.57	.53	.20	.57
55	.62	.16	.12	.06	.15
56	.38	.57	.52	.22	.56
57	.64	.79	.75	.30	.74
58	.28	.65	.61	.22	.65

(Cont'd)

TABLE 64 (Cont'd)

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
59	.83	.61	.57	.16	.30
60	.87	.87	.84	.19	.38
61	.77	.65	.61	.20	.46
62	.73	.63	.59	.21	.51
63	.74	.76	.72	.25	.61
64	.93	.87	.84	.12	.21
65	.88	.51	.47	.10	.15
66	.93	.94	.91	.12	.23
67	.93	1.02	.99	.14	.23
68	.78	.78	.74	.23	.51
69	.75	.63	.59	.20	.42
70	.73	.49	.45	.16	.40
71	.81	.59	.55	.16	.26
72	.85	.84	.80	.20	.44
73	.77	.76	.73	.23	.48
74	.62	.76	.72	.29	.70
75	.64	.87	.83	.32	.82
76	.69	.86	.82	.30	.77
77	.72	.94	.90	.31	.71
78	.67	.81	.77	.29	.68
79	.83	.34	.30	.09	.11
80	.80	.38	.34	.11	.22
81	.81	.53	.50	.15	.33
82	.34	.60	.56	.22	.58
83	.76	.61	.57	.19	.48

$n = 137$ $S^2 = 229.71$ KR-20 = .95

$\overline{X} = 58.96$ $S = 15.16$ $S.E._{meas} = 3.41$

TABLE 65. ITEM ANALYSIS FOR FRACTION ACHIEVEMENT TEST GRADE EIGHT

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
1	.99	.57	.54	.01	.03
2	.62	.70	.65	.27	.70
3	.93	.44	.40	.06	.13
4	.64	.67	.61	.25	.72
5	.90	.37	.32	.07	.23
6	.84	.64	.59	.16	.36
7	.84	.70	.65	.17	.38
8	.59	.79	.73	.31	.80
9	.71	.80	.75	.28	.67
10	.64	.77	.72	.29	.77
11	.93	.87	.83	.12	.26
12	.90	.70	.65	.12	.18
13	.94	.54	.50	.07	.11
14	.94	.94	.90	.11	.23
15	.90	.95	.91	.16	.28
16	.94	.69	.65	.08	.16
17	.88	.59	.55	.12	.31
18	.88	.81	.76	.16	.34
19	.88	.82	.78	.16	.36
20	.73	.41	.36	.14	.29
21	.68	.67	.62	.24	.55
22	.84	.59	.54	.14	.31
23	.90	.71	.66	.12	.26
24	.88	.76	.72	.16	.33
25	.74	.66	.60	.21	.47
26	.95	.86	.82	.10	.15
27	.93	.57	.53	.08	.11
28	.74	.59	.54	.19	.42
29	.82	.90	.85	.24	.49

(Cont'd)

TABLE 65 (Cont'd)

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
30	.63	.65	.59	.24	.57
31	.63	.47	.41	.18	.42
32	.86	.24	.19	.05	.11
33	.82	.65	.60	.17	.44
34	.72	.76	.71	.27	.64
35	.73	.60	.55	.20	.52
36	.80	.77	.72	.22	.49
37	.57	.59	.53	.23	.60
38	.93	.59	.54	.08	.15
39	.68	.61	.55	.22	.49
40	.85	.67	.63	.16	.31
41	.86	.76	.72	.17	.36
42	.71	.73	.67	.25	.57
43	.66	.76	.70	.28	.72
44	.62	.64	.58	.24	.62
45	.92	.43	.39	.07	.13
46	.72	.70	.65	.24	.57
47	.80	.54	.48	.15	.39
48	.86	.26	.22	.06	.18
49	.80	.14	.09	.04	.16
50	.69	.85	.79	.30	.69
51	.83	.93	.88	.24	.51
52	.74	.62	.57	.20	.56
53	.93	.56	.51	.07	.15
54	.82	.48	.43	.13	.36
55	.84	.61	.57	.15	.39
56	.91	.79	.75	.13	.31
57	.95	.41	.37	.04	.10
58	.93	.88	.83	.12	.23

(Cont'd)

TABLE 65 (Cont'd)

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
59	.86	.59	.54	.13	.23
60	.87	.81	.76	.17	.38
61	.80	.72	.67	.21	.56
62	.85	.72	.67	.17	.39
63	.87	.53	.49	.11	.21
64	.70	.66	.60	.23	.59
65	.83	.66	.61	.17	.38
66	.51	.62	.56	.25	.68
67	.92	.85	.81	.13	.26
68	.74	.65	.60	.21	.52

$n = 146$ $S^2 = 125.88$ KR-20 = .94

$\overline{X} = 54.79$ $S = 11.22$ $S.E._{meas} = 2.84$

TABLE 66. ITEM ANALYSIS FOR FRACTION RETENTION TEST GRADE FOUR

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
1	.96	.87	.82	.07	.14
2	.64	.83	.76	.31	.75
3	.64	.86	.79	.32	.79
4	.64	.78	.71	.29	.68
5	.83	.75	.69	.19	.43
6	.79	.56	.49	.16	.33
7	.88	.91	.85	.18	.36
8	.87	.88	.81	.18	.39
9	.82	.85	.79	.23	.57
10	.59	.88	.81	.34	.82
11	.53	.75	.67	.30	.79
12	.51	.88	.81	.35	.90
13	.88	.88	.82	.18	.39
14	.61	.92	.85	.35	.82
15	.74	.48	.41	.16	.33
16	.51	.72	.65	.29	.68
17	.66	.73	.65	.27	.71
18	.62	.86	.79	.33	.89
19	.45	.78	.71	.31	.86
20	.39	.81	.74	.31	.79
21	.95	.78	.72	.08	.14
22	.94	.78	.72	.10	.21
23	.92	.94	.89	.14	.25
24	.91	.57	.51	.09	.21
25	.91	.61	.55	.10	.25
26	.93	.67	.62	.09	.18
27	.92	.63	.57	.10	.18
28	.91	.78	.72	.13	.25
29	.91	.78	.72	.13	.25

(Cont'd)

TABLE 66 (Cont'd)

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
30	.91	.85	.79	.14	.29
31	.88	.85	.78	.17	.36
32	.69	.54	.47	.19	.54
33	.66	.57	.50	.21	.57
34	.24	.60	.53	.18	.59
35	.21	.72	.65	.21	.62
36	.58	.64	.57	.25	.72
37	.23	.43	.36	.13	.34
38	.53	.51	.44	.20	.40
39	.06	.28	.22	.04	.10
40	.51	.59	.52	.24	.75
41	.44	.59	.52	.23	.62
42	.16	.52	.46	.13	.38
43	.28	.30	.23	.10	.27

$n = 109$ $S^2 = 72.08$ KR-20 = .92

$\overline{X} = 28.21$ $S = 8.49$ $S.E._{meas} = 2.35$

TABLE 67. ITEM ANALYSIS FOR FRACTION RETENTION TEST GRADE SIX

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
1	.93	.50	.47	.07	.14
2	.92	.20	.17	.03	.03
3	.64	.33	.30	.13	.25
4	.76	.63	.60	.20	.53
5	.91	.74	.71	.12	.25
6	.91	.48	.45	.08	.17
7	.89	.49	.46	.09	.19
8	.83	.68	.65	.17	.36
9	.60	.64	.61	.25	.61
10	.95	.59	.56	.06	.14
11	.64	.81	.77	.30	.78
12	.30	.72	.68	.25	.72
13	.34	.73	.69	.26	.78
14	.39	.60	.56	.23	.64
15	.41	.64	.61	.25	.69
16	.47	.63	.59	.25	.67
17	.90	.96	.93	.17	.33
18	.95	.84	.81	.08	.17
19	.85	.83	.80	.19	.36
20	.77	.86	.83	.26	.64
21	.72	.74	.70	.25	.58
22	.73	.75	.71	.25	.58
23	.88	.88	.85	.18	.42
24	.91	.83	.80	.13	.28
25	.84	.77	.74	.19	.42
26	.94	1.08	1.05	.13	.22
27	.73	.67	.63	.22	.56
28	.81	.85	.82	.23	.53
29	.72	.64	.61	.22	.53

(Cont'd)

TABLE 67 (Cont'd)

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
30	.77	.77	.73	.23	.50
31	.83	.93	.89	.24	.58
32	.77	.82	.78	.25	.56
33	.46	.69	.65	.27	.75
34	.53	.68	.64	.27	.75
35	.64	.93	.89	.35	.89
36	.63	.87	.83	.33	.89
37	.64	.87	.84	.33	.92
38	.68	.85	.81	.30	.78
39	.80	.64	.60	.18	.44
40	.58	.59	.55	.23	.61
41	.73	.38	.35	.13	.31
42	.45	.54	.50	.21	.53
43	.35	.65	.61	.24	.67
44	.36	.60	.66	.22	.58
45	.89	.79	.76	.15	.31
46	.92	.12	.09	.02	.06
47	.82	.25	.21	.07	.17
48	.96	.59	.56	.05	.06
49	.95	.11	.09	.01	.0
50	.90	.51	.48	.09	.19
51	.87	.52	.49	.11	.22
52	.63	.50	.47	.19	.47
53	.99	.35	.33	.01	.03
54	.88	.37	.34	.08	.19
55	.54	.72	.68	.29	.75
56	.40	.67	.63	.26	.75
57	.39	.74	.70	.28	.78
58	.35	.66	.62	.25	.69

(Cont'd)

TABLE 67 (Cont'd)

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
59	.41	.62	.59	.24	.69
60	.42	.55	.52	.22	.61
61	.84	.90	.86	.22	.47
62	.88	.85	.82	.17	.36
63	.79	.84	.80	.24	.58
64	.82	.75	.72	.20	.42
65	.79	.81	.78	.24	.58
66	.93	.71	.68	.10	.19
67	.92	.71	.68	.11	.22
68	.92	.99	.96	.15	.28
69	.88	.82	.79	.16	.36
70	.79	.80	.77	.23	.56
71	.68	.61	.58	.22	.61
72	.64	.52	.48	.19	.47
73	.80	.73	.69	.21	.47
74	.73	.62	.58	.20	.44
75	.79	.84	.80	.24	.58
76	.70	.88	.85	.31	.78
77	.62	.78	.75	.30	.78
78	.68	.93	.89	.33	.86
79	.71	.89	.86	.31	.78
80	.63	.81	.77	.31	.83
81	.72	.71	.68	.24	.56
82	.83	.59	.56	.15	.36
83	.75	.63	.59	.20	.50
84	.45	.70	.66	.28	.72
85	.80	.63	.60	.18	.39

(Cont'd)

TABLE 67 (Cont'd)

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
86	.50	.64	.60	.25	.64
87	.64	.51	.48	.19	.47

n = 137 S^2 = 296.83 KR-20 = .96

$\overline{X}$ = 62.33 S = 17.23 $S.E._{meas}$ = 3.39

TABLE 68. ITEM ANALYSIS FOR FRACTION RETENTION TEST GRADE EIGHT

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
1*	1.00	0.0	0.0	0.0	0.0
2	.80	.66	.59	.19	.43
3	.82	.27	.20	.07	.12
4	.96	.39	.33	.03	.11
5	.94	.35	.29	.04	.16
6	.74	.75	.68	.24	.66
7	.82	.89	.82	.23	.50
8	.81	.92	.85	.25	.53
9	.94	.55	.49	.07	.16
10	.82	.75	.68	.20	.47
11	.90	.89	.82	.16	.27
12	.95	.72	.67	.07	.13
13	.94	.45	.39	.06	.08
14	.97	.69	.64	.05	.06
15	.90	.65	.58	.11	.16
16	.91	.60	.53	.10	.21
17	.88	.60	.53	.12	.29
18	.86	.58	.52	.13	.24
19	.49	.29	.22	.12	.47
20	.84	.66	.59	.16	.42
21	.75	.49	.41	.15	.37
22	.91	.38	.31	.06	.08
23	.95	.48	.43	.05	.06
24	.92	.67	.60	.10	.18
25	.80	.57	.50	.16	.30
26	.95	1.07	1.02	.11	.18
27	.86	.70	.64	.15	.29
28	.97	1.01	1.05	.08	.08
29	.93	.65	.59	.09	.16

(Cont'd)

TABLE 68 (Cont'd)

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
30	.78	.73	.66	.22	.58
31	.87	.70	.64	.15	.29
32	.76	.52	.45	.16	.40
33	.91	.30	.24	.05	.09
34	.88	.35	.29	.07	.19
35	.81	.55	.48	.15	.32
36	.77	.50	.42	.15	.33
37	.80	.66	.59	.19	.38
38	.66	.61	.54	.23	.51
39	.97	.62	.56	.05	.11
40	.71	.51	.44	.18	.48
41	.90	.54	.48	.09	.16
42	.72	.67	.59	.22	.53
43	.72	.64	.57	.22	.51
44	.66	.62	.55	.23	.54
45	.92	.01	.05	.00	-.02
46	.88	.56	.49	.11	.19
47	.38	.48	.41	.18	.54
48	.99	.28	.24	.01	.03
49	.86	.75	.68	.17	.42
50	.93	.66	.60	.09	.24
51	.61	.76	.68	.29	.79
52	.84	.88	.81	.21	.50
53	.67	.72	.64	.26	.77
54	.90	.85	.79	.15	.26
55	.97	.72	.67	.05	.08
56	.97	.82	.76	.06	.11
57	.96	.63	.58	.06	.08
58	.92	.95	.89	.14	.26

(Cont'd)

TABLE 68 (Cont'd)

Item	Difficulty	Biserial Correlation	Corrected Biserial Correlation	Reliability	Discriminating Power
59	.96	.91	.89	.10	.16
60	.95	.90	.84	.10	.16
61	.91	.82	.76	.13	.21
62	.88	.83	.77	.16	.34
63	.67	.29	.21	.10	.35

*Deleted from factor analysis.

n = 146 $S^2 = 65.09$ KR-20 = .90

$\overline{X}$ = 53.40 S = 8.07 $S.E._{meas} = 2.51$

APPENDIX C

TABLE 69. PRINCIPAL FACTOR ANALYSIS OF TBMC (TEN FACTOR SOLUTION) GRADE FOUR

	Factor										Communal-ities
Items	1	2	3	4	5	6	7	8	9	10	h^2
1							.72*				.58
2							.36*				.49
3					.74*						.59
4					.61*						.71
5					.69*						.56
6		.48*									.60
7		.38*	.42*						.35*		.60
8	.54										.48
9											.62
10										.55	.58
11								.57			.69
12		.77*									.75
13		.73*									.59
14	.61*										.57
15		.48*						.46*			.59
16	.65*										.57
17	.62*										.56
18	.76*										.70
19									.68		.53
20				.78*							.69
21			.42				.46				.56
22		.35*		.40*		.38*					.64
23				.78*							.73
24			.39*			.38*					.51
25											.56

(Cont'd)

TABLE 69 (Cont'd)

Items	Factor 1	2	3	4	5	6	7	8	9	10	Communalities h^2
26						.51		.35			.54
27											.57
28									.72		.56
29											.50
30											.43
% Total variance	8.27	7.31	6.43	6.03	5.78	5.60	4.94	4.89	4.79	4.65	
% Common variance	14.09	12.45	10.96	10.27	9.87	9.54	8.42	8.34	8.15	7.91	
Sum of communalities											17.61

Total variance accounted for 58.70%

*Items included on subtests

TABLE 70. PRINCIPAL FACTOR ANALYSIS OF TBMC (TEN FACTOR SOLUTION) GRADE SIX

	Factors										Communal- ities
Items	1	2	3	4	5	6	7	8	9	10	h^2
1			.43*						.51*		.60
2						.48*					.49
3			.60*								.59
4			.47*								.47
5			.43*		.47*						.57
6			.40*	.63*							.65
7				.64*		.35*					.62
8	.59										.52
9											.54
10										.63	.52
11										.52	.56
12		.68*									.50
13		.73*									.73
14	.76*										.64
15		.58*		.39*							.61
16	.58*										.57
17									.70*		.58
18	.74*										.64
19				.66*							.61
20					.72*						.62
21						.59*					.51
22		.39*									.39
23											.61
24						.66*					.51
25										.61	.66

(Cont'd)

TABLE 70 (Cont'd)

Items	Factors 1	2	3	4	5	6	7	8	9	10	Communal-ities h^2
26							.80				.67
27								.80			.68
28					.65						.57
29								.81			.70
30							.69				.61
% Total variance	7.29	6.56	6.15	6.07	5.89	5.23	5.20	5.11	5.01	4.95	
% Common variance	12.68	11.42	10.70	10.57	10.24	9.11	9.05	8.90	8.73	8.61	
Sum of communalities											18.38

Total variance accounted for 57.45%

*Items included on subtests

TABLE 71. PRINCIPAL FACTOR ANALYSIS OF TBMC (TEN FACTOR SOLUTION) GRADE EIGHT

	Factors										Communal-ities
Items	1	2	3	4	5	6	7	8	9	10	h^2
1				.59*							.52
2				.43*	.40*						.48
3											.50
4					.67*					.38*	.65
5					.75*						.72
6		.39				.47					.52
7		.69*									.69
8									.67		.65
9									.74		.61
10								.68			.62
11								.66			.52
12			.34			.47*					.53
13					.38	.53					.52
14	.75*										.60
15	.33*										.30
16	.68*										.53
17	.66*										.56
18	.81*										.75
19		.55*									.50
20			.34*	.64*							.62
21		.55*		.36*							.59
22		.41*									.47
23		.57*									.55
24		.42*									.51
25				.37							.53

(Cont'd)

TABLE 71 (Cont'd)

Items	Factors 1	2	3	4	5	6	7	8	9	10	Communal-ities h^2
26			.76*								.68
27							.77				.63
28								.59			.62
29							.74				.68
30			.81*								.74
% Total variance	8.35	7.03	6.21	5.97	5.89	5.60	5.33	4.97	4.94	3.89	
% Common variance	14.34	12.08	10.68	10.26	10.13	9.62	9.17	8.55	8.49	6.69	
Sum of communalities											18.62

Total variance accounted for 58.18%

*Items included on subtests

TABLE 72. PRINCIPAL FACTOR ANALYSIS OF FRACTION ACHIEVEMENT TEST (TEN FACTOR SOLUTION) GRADE FOUR

	Factors										Communal-ities
Items	1	2	3	4	5	6	7	8	9	10	h^2
1		.81*									.79
2	.85*										.82
3		.37*							.36*		.62
4	.83*										.76
5		.70									.63
6		.49*							.59*		.71
7		.84*									.81
8		.83									.77
9		.47*			.56*						.66
10	.36										.74
11	.76*										.71
12	.58					.38					.58
13		.43*			.49*						.61
14						.76					.68
15	.78*										.78
16	.50					.64					.78
17	.83*										.76
18	.72										.67
19	.57*					.39*					.69
20	.55					.51					.72
21									.76*		.71
22					.53*						.56
23		.37*	.46*								.66
24					.69*						.64
25					.65*						.68

(Cont'd)

TABLE 72 (Cont'd)

Items	Factors 1	2	3	4	5	6	7	8	9	10	Communal-ities h^2
26			.60*						.43*		.63
27			.77*								
28			.83*								
29			.83*								
30		.35*	.54*								
31		.34*	.54*								
32							.88*				
33							.87*				
34								.90			
35								.89*			.89
36	.47*									.47*	.62
37				.33*							.43
38	.36*						.59*			.53*	.66
39				.89*							.82
40	.56*										.49
41	.47*							.33*			.50
42				.83*							.75
43				.89*							.85
% Total variance	15.78	9.57	8.87	6.64	5.85	5.72	5.65	5.27	4.43	3.74	
% Common variance	22.06	13.38	12.40	9.24	8.18	8.00	7.90	7.37	6.20	5.23	
Sum of communalities											30.75

Total variance accounted for 71.52%

*Items included on subtests

TABLE 73. PRINCIPAL FACTOR ANALYSIS OF FRACTION ACHIEVEMENT TEST (FIFTEEN FACTOR SOLUTION) GRADE SIX

	Factors															Communal-ities
Items	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	h^2
1													.78*			.70
2													.68*			.68
3													.63*			.59
4														.63		.56
5				.72*												.68
6														.59		.54
7												.47				.43
8				.43*										.52*		.65
9				.51*												.48
10				.53*										.44*		.68
11				.87*												.84
12				.81*												.73
13				.72*												.61
14											.88					.82
15											.89					.85
16											.63					.52
17												.49				.44
18										.48						.59
19		.76*								.36*						.76
20		.73*								.33*						.72
21		.77*														.75
22		.50*														.51
23		.75*														.76
24		.73*														.67
25												.59			.36	.70

(Cont'd)

TABLE 73 (Cont'd)

Items	Factors 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Communal- ities h^2
26												.57			.39	.76
27		.81*														.77
28		.79*														.75
29								.47*								.62
30								.53*								.57
31								.61*								.70
32								.62*								.64
33								.78*								.76
34								.77*								.76
35			.37								.34					.61
36			.35							.55						.57
37	.41									.42						.57
38	.37															.56
39																.46
40			.78*													.75
41			.67*													.66
42			.68*													.55
43			.81*													.75
44			.46*													.61
45			.36*						.41*							.43
46			.48*													.65
47			.53*													.59
48						.73*										.75
49						.79*										.79
50						.78*										.74

(Cont'd)

TABLE 73 (Cont'd)

Items	Factors 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Communal-ities h^2
51																.46
52						.72*										.62
53						.41*										.45
54	.41*								.41*			.34*				.65
55	.62*											.39*				.83
56	.57*											.41*				.80
57	.36*								.67*							.74
58	.42*								.68*							.75
59	.44*								.68*							.73
60	.76*															.69
61	.87*															.87
62	.88*															.77
63	.62*															.59
64	.76*															.80
65	.77*															.81
66	.81*															.80
67	.73*															.72
68	.75*														.37*	.78
69	.72*														.44*	.81
70															.62*	.49
71	.45*												.36*		.51*	.72
72					.82*											.74
73					.87*											.80
74					.76*											.74
75					.41*					.53*						.63

(Cont'd)

TABLE 73 (Cont'd)

Items	Factors 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Communal-ities h^2
76					.36*					.56*						.62
77					.66*											.58
78							.59*									.55
79							.63*									.59
80							.74*									.69
81							.44*			.46*						.60
82							.66*									.71
83							.61*			.38*						.67
% Total variance	11.50	6.22	5.44	5.15	4.31	4.25	4.14	4.06	3.53	3.51	3.32	3.15	3.14	2.57	2.48	
% Common variance	17.22	9.31	8.15	7.71	6.45	6.36	6.20	6.09	5.29	5.26	4.97	4.71	4.71	3.84	3.71	
Sum of communalities																55.41

Total variance accounted for 66.75%

*Items included on subtests

TABLE 74. PRINCIPAL FACTOR ANALYSIS OF FRACTION ACHIEVEMENT TEST (FIFTEEN FACTOR SOLUTION) GRADE EIGHT

Items	Factors															Communal-ities
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	h^2
1											.74*					.67
2							.44*								.33*	.52
3															.46*	.55
4																.50
5											.57*	.34*	.35*			.67
6							.35*						.35			.65
7	.50															.54
8						.83										.72
9						.81										.72
10						.75										.59
11							.69*									.69
12	.46*															.57
13	.51*						.36*									.75
14	.48*						.35*									.71
15	.76*															.75
16	.84*															.77
17	.82*															.82
18							.66*									.76
19	.76*															.71
20	.68*											.34*				.73
21								.43								.59
22												.60				.68
23			.37					.43								.53
24												.72				.72
25										.72						.64

(Cont'd)

TABLE 74 (Cont'd)

Items	Factors 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Communal-ities h^2
26										.69						.65
27										.47		.33				.67
28					.60								.39			.70
29		.34					.37									.68
30			.43		.37											.61
31					.70*											.71
32					.44*				.52*							.71
33									.77*							.72
34									.66*							.69
35					.73*											.79
36					.69*											.77
37											.38					.59
38													.70			.58
39													.55			.68
40														.65		.65
41															.46	.62
42		.66*														.63
43		.72*														.62
44		.32*														.52
45		.70*														.71
46		.67*														.72
47		.64*														.71
48		.71*														.64
49		.70*														.67
50								.73*								.73
51								.76*								.77
52				.57*												.62

(Cont'd)

TABLE 74 (Cont'd)

Items	Factors 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Communal-ities h^2
53		.49*														.59
54				.59*												.56
55																.59
56																.46
57	.34*		.60*													.61
58				.56*												.57
59				.61*												.57
60			.38*	.56*												.64
61				.61*												.52
62			.63*													.56
63			.58*													.47
64			.71*													.64
65			.65*													.61
66			.60*	.39*												.65
67			.53*	.49*												.62
68			.52*	.36*												.56
% Total variance	7.81	7.27	6.69	5.66	4.66	3.97	3.95	3.71	3.58	3.18	3.05	3.01	2.91	2.72	2.42	
% Common variance	12.08	11.26	10.36	8.76	7.21	6.14	6.12	5.75	5.54	4.92	4.73	4.67	4.50	4.22	3.75	
Sum of communalities																43.92

Total variance accounted for 64.59%

*Items included on subtests

TABLE 75. PRINCIPAL FACTOR ANALYSIS OF FRACTION RETENTION TEST (TEN FACTOR SOLUTION) GRADE FOUR

Items	Factors										Communal-ities h^2
	1	2	3	4	5	6	7	8	9	10	
1		.46*					.49*				.69
2	.80*										.75
3	.83*										.79
4	.81*										.72
5		.78									.69
6		.78*									.80
7		.79*									.75
8		.78									.69
9	.38	.61									.67
10	.54				.53						.69
11	.74*										.65
12	.65				.38						.64
13		.71*									.68
14	.56				.54						.65
15		.59*			.45*						.62
16					.66						.65
17	.76*										.72
18	.66										.57
19	.71*										.68
20	.66										.63
21				.55*							.54
22			.79*								.70
23				.34*							.52
24			.86*								.77
25			.93*								.88

(Cont'd)

TABLE 75 (Cont'd)

Items	Factors 1	2	3	4	5	6	7	8	9	10	Communal-ities h^2
26				.86*							.81
27				.81*							.75
28				.41*			.81*				.91
29				.41*			.81*				.91
30			.53*	.43*							.66
31			.54*	.34*							.67
32						.92					.91
33						.91					.91
34									.84		.85
35									.80		.80
36					.39*						.44
37					.49*						.53
38					.63*						.49
39									.36*	.60*	.66
40											.70
41											.77
42										.54*	.58
43										.83*	.79
% Total variance	15.20	10.57	7.77	6.72	6.10	5.31	5.29	4.71	4.65	4.08	
% Common variance	21.59	15.01	11.04	9.55	8.66	7.54	7.51	6.69	6.61	5.80	
Sum of communalities											30.27

Total variance accounted for 70.40%

*Items selected for inclusion in subtests

TABLE 76. PRINCIPAL FACTOR ANALYSIS OF FRACTION RETENTION TEST (FIFTEEN FACTOR SOLUTION) GRADE SIX

Item	Factors															Communal-ities
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	h^2
1											.59*	.34*				.64
2							.40	.35					.33			.68
3											.52*					.58
4														.75		.67
5							.40*									.43
6														.74		.63
7	.36												.59			.66
8				.34									.44*		.68	.73
9					.50											.67
10															.52*	.55
11							.74*	.37*								.83
12							.78*									.74
13							.65									.72
14							.49*								.44*	.70
15					.41		.44									.64
16							.60*									.60
17							.53		.37							.66
18											.63					.58
19				.37							.54					.66
20											.61					.60
21		.34*											.41*			.66
22		.36*					.34*						.37*			.64
23		.84*														.78
24		.70*														.65
25		82*														.78

(Cont'd)

TABLE 76 (Cont'd)

Items	Factors															Communal-ities h^2
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
26		.42*														.52
27		.82*														.72
28		.80*														.71
29		.87*														.81
30		.87*														.81
31		.82*														.74
32		.86*														.78
33					.84*											.89
34					.60*											.64
35					.61*											.62
36					.75*											.75
37					.66*											.70
38					.37*			.45*								.55
39																.67
40								.38								.60
41												.58				.65
42																.54
43												.44	.42			.65
44								.78*								.68
45								.61*								.70
46								.75*								.73
47								.64*								.67
48					.34*					.68*						.81
49	.33*									.55*						.57
50										.71*						.70

(Cont'd)

TABLE 76 (Cont'd)

Items	Factors 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Communal-ities h^2
51										.76*						.87
52									.72*							.71
53									.81*							.77
54									.78*							.73
55				.36*					.40*							.64
56									.64*							.54
57																.45
58			.71*													.72
59			.69*													.72
60			.75*													.71
61	.41*		.75*													.84
62	.36*		.70*													.77
63			.68*													.73
64	.69*															.70
65	.63*															.55
66	.65*															.59
67	.56*															.55
68	.68*															.76
69	.69*															.72
70	.80*															.77
71	.71*															.69
72	.81*															.76
73	.77*															.75
74	.40*		.44*													.53
75	.74*															.73

(Cont'd)

TABLE 76 (Cont'd)

Items	Factors 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Communal-ities h^2
76				.92*												.89
77				.86*												.85
78				.73*												.72
79				.56*		.41*										.68
80				.33*		.47*						.37*				.61
81				.72*												.66
82				.48*												.62
83						.72*										.67
84						.76*										.75
85						.73*										.69
86						.59*										.57
87						.73*										.71
% Total variance	10.07	8.73	5.46	5.30	5.15	4.62	4.56	4.53	4.23	3.72	2.88	2.53	2.28	2.09	2.05	
% Common variance	14.76	12.81	8.00	7.77	7.55	6.77	6.68	6.64	6.21	5.45	4.23	3.70	3.34	3.07	3.01	
Sum of communalities																59.32

Total variance accounted for 68.18%

*Items included in subtests

TABLE 77. PRINCIPAL FACTOR ANALYSIS OF FRACTION RETENTION TEST (FIFTEEN FACTOR SOLUTION) GRADE EIGHT

Items	Factors															Communal-ities
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	h^2
1									.55*							.40
2	.33*						.68*		.69*							.75
3							.81*									.77
4							.63*									.52
5											.48*					.59
6																.58
7										.65						.51
8										.66						.70
9	.47*															.66
10	.67*															.68
11										.68*						.74
12	.63*															.61
13	.74*															.65
14	.73*															.69
15	.52*									.41*						.69
16			.45*													.60
17	.82*															.76
18	.63*															.63
19	.36							.36								.52
20		.41														.69
21		.69														.70
22		.43							.55							.68
23		.68														.74
24		.65														.56
25		.78														.72

(Cont'd)

TABLE 77 (Cont'd)

Items	Factors 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Communal-ities h^2
26		.50														.67
27											.47					.67
28																.56
29			.49*						.57*							.78
30			.39*						.40*							.53
31			.68*								.34*					.74
32			.61*								.48*					.73
33			.79*													.71
34			.74*													.69
35	.50															.60
36												.75				.67
37												.72				.64
38	.42										.36					.58
39														.36		.57
40													.40*	.34*		.47
41						.44*								.40*		.63
42						.61*										.54
43						.54*										.65
44				.51*	.34*											.64
45				.40*												.62
46				.78*												.74
47				.80*												.70
48				.36*									.38			.55
49				.55*									.35			.56
50				.54*												.42

(Cont'd)

TABLE 77 (Cont'd)

Items	Factors 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Communal-ities h^2
51															.78	.68
52														.70*		.57
53												.34*		.47*		.55
54					.47*						.43*					.55
55					.54*										.42*	.65
56					.39*											.52
57								.73*								.64
58											.66*					.67
59								.65*								.70
60					.55*											.58
61					.74											.68
62					.70											.66
% Total variance	8.30	5.56	5.32	5.29	4.53	4.28	3.97	3.71	3.63	3.47	3.25	3.14	3.13	2.88	2.78	
% Common variance	12.13	8.80	8.41	8.37	7.16	6.77	6.28	5.87	5.74	5.49	5.14	4.96	4.94	4.55	4.40	
Sum of communalities																39.21

Total variance accounted for 63.23%

*Items included on subtests

APPENDIX D

TABLE 78. CORRELATION COEFFICIENTS (PPM) AMONG THE TBMC AND FRACTION ACHIEVEMENT SUBTESTS GRADE FOUR

		TBMC Subtests				Fraction Achievement Subtests			
		1	2	3	4	1	2	3	4
TBMC	1	1.0							
	2	.19	1.0						
	3	.13	.35	1.0					
	4	.10	.21	.09	1.0				
Fraction	1	.16	.31	.16	.13	1.0			
	2	.06	.28	.30	.17	.10	1.0		
	3	.18	.30	.28	.04	.43	.26	1.0	
	4	.08	.20	.23	.19	.19	.67	.30	1.0

TABLE 79. CORRELATION COEFFICIENTS (PPM) AMONG THE TBMC AND FRACTION ACHIEVEMENT SUBTESTS GRADE SIX

		TBMC Subtests 1	2	3	Fraction Achievement Subtests 1	2	3	4	5	6	7	8
TBMC	1	1.0										
	2	.29	1.0									
	3	.23	.27	1.0								
Fractions	1	.19	.27	.32	1.0							
	2	.13	.19	.16	.22	1.0						
	3	.29	.26	.26	.39	.31	1.0					
	4	.17	.24	.31	.28	.23	.48	1.0				
	5	.26	.21	.32	.25	.31	.43	.48	1.0			
	6	.28	.23	.30	.23	.20	.42	.52	.47	1.0		
	7	.01	.06	.28	.14	.32	.20	.34	.27	.26	1.0	
	8	.09	.06	.33	.26	.24	.25	.32	.27	.47	.43	1.0

TABLE 80. CORRELATION COEFFICIENTS (PPM) AMONG THE TBMC AND FRACTION ACHIEVEMENT SUBTESTS GRADE EIGHT

		TBMC Subtests 1	TBMC Subtests 2	TBMC Subtests 3	Fraction Achievement Subtests 1	Fraction Achievement Subtests 2	Fraction Achievement Subtests 3	Fraction Achievement Subtests 4	Fraction Achievement Subtests 5
TBMC	1	1.0							
	2	.26	1.0						
	3	.25	.35	1.0					
Fractions	1	.26	.33	.28	1.0				
	2	.27	.26	.32	.47	1.0			
	3	.14	.26	.33	.46	.43	1.0		
	4	.19	.25	.27	.36	.47	.42	1.0	
	5	.33	.21	.26	.45	.35	.35	.47	1.0

TABLE 81. CORRELATION COEFFICIENTS (PPM) AMONG THE TBMC AND FRACTION RETENTION SUBTESTS GRADE FOUR

		TBMC Subtests				Fraction Achievement Subtests			
		1	2	3	4	1	2	3	4
TBMC	1	1.0							
	2	.19	1.0						
	3	.13	.35	1.0					
	4	.10	.21	.09	1.0				
Fraction	1	.19	.40	.20	.16	1.0			
	2	.24	.38	.38	.28	.34	1.0		
	3	.04	.23	.31	.04	.26	.26	1.0	
	4	.07	.30	.26	.16	.27	.36	.33	1.0

TABLE 82. CORRELATION COEFFICIENTS (PPM) AMONG THE TBMC AND FRACTION RETENTION SUBTESTS GRADE SIX

		TBMC Subtests 1	2	3	Fraction Retention Subtests 1	2	3	4	5	6	7	8
TBMC	1	1.0										
	2	.29	1.0									
	3	.23	.27	1.0								
Fractions	1	.20	.30	.30	1.0							
	2	.22	.18	.22	.35	1.0						
	3	.29	.37	.26	.35	.39	1.0					
	4	.17	.23	.26	.48	.34	.65	1.0				
	5	.39	.26	.31	.25	.41	.42	.55	1.0			
	6	.25	.30	.34	.25	.39	.56	.54	.54	1.0		
	7	.17	.11	.17	.08	.23	.24	.26	.37	.44	1.0	
	8	.30	.11	.23	.16	.29	.30	.31	.42	.48	.55	1.0

TABLE 83. CORRELATION COEFFICIENTS (PPM) AMONG THE TBMC AND FRACTION RETENTION SUBTESTS GRADE EIGHT

		TBMC Subtests			Fraction Retention Subtests				
		1	2	3	1	2	3	4	5
TBMC	1	1.0							
	2	.26	1.0						
	3	.25	.35	1.0					
Fraction	1	.10	.26	.29	1.0				
	2	.30	.30	.27	.37	1.0			
	3	.15	.10	.20	.38	.33	1.0		
	4	.17	.10	.22	.18	.35	.30	1.0	
	5	.33	.20	.13	.25	.47	.27	.49	1.0

APPENDIX E

Observation Schedule — Definition of Terms

time (left hand side): The time of introduction of a model into instruction is recorded once.

Model: Linear - number line
Area - any region diagram which has been divided into parts or physical representation such as: sheet of paper; chocolate bar; felt board with figures.

Media: indicates medium being used to illustrate the model, e.g., blackboard, overhead projector, etc.

Initiator: Each statement or question associated with the model is initiated by the teacher ( ) or a student ( ).

Type: discourse associated with each model.
expos. - indicates the initiator is explaining or making a statement of fact.
inq. - indicates a question is being asked.

Response Representation: enactive - if the response involves manipulation of materials, e.g., paper folding.
iconic - if the response involves drawing a diagram or selecting a diagram.
symbolic - if the response is verbal and is not accompanied by either of the above.

Response Level: recall - the simplest level of response, 1, involves memory, e.g., given a square which is divided into four parts: "How many parts are there?" The response is level 1. Responses which require rule using are placed in level 2, and responses requiring problem solving and application are placed in level 3.

time (right hand side): When attention is moved from the model to a different model or to abstract discussion.

School______________________ Teacher______________________ Grade______

Date__________ Observer__________ Group Size________ Time__________to________

time	Model		Media	Initiator		Type		Response Representation			Response Level			time
	Linear	Area		Teacher	Student	Expos.	Inq.	Enactive	Iconic	Symbolic	1	2	3	
											(Recall)		(Application)	
1.														
2.														
3.														
4.														
5.														
6.														
7.														
8.														
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